# Hilbert properties of varieties

Arno Fehm[1]

AHGT Seminar
February 2024

[1] Technische Universität Dresden

# History Part 1: Around 1900

For the whole talk, $K$ is a field of characteristic $0$, $G$ a finite group.

**Theorem (Hilbert 1892, HIT)**

*If $K$ is a number field and $f \in K[t,x] \setminus K[t]$ irreducible, there exist infinitely many $\tau \in K$ with $f(\tau, x) \in K[x]$ irreducible.*

This then also holds for $f_1, \ldots, f_r \in K[t_1, \ldots, t_n, x_1, \ldots, x_m]$.
Quantitative versions and other variants exist.

**Corollary**

*If $K$ is a number field and $g \in K(t)[x]$ irreducible, there exist $\tau \in K$ with $\mathrm{Gal}(g(t,x)/K(t)) \cong \mathrm{Gal}(g(\tau,x)/K)$.*

**Proof.**

Apply HIT to minimal polynomial $f$ of integral generator of splitting field of $g$, discard finitely many $\tau \in K$. □

# History Part 1: Around 1900

**Corollary**

*Let $G \leq S_n$ act on $\mathbb{Q}(z_1, \ldots, z_n)$ by permuting the $z_i$.*
*If $\mathbb{Q}(\underline{z})^G$ is rational, there exists $L/\mathbb{Q}$ Galois with $\mathrm{Gal}(L/\mathbb{Q}) \cong G$.*

This way, Hilbert realized $S_n$ and $A_n$ as Galois groups over $\mathbb{Q}$.

**Question (Noether)**

*Is $\mathbb{Q}(\underline{z})^G$ rational for every finite group $G$?*
*Equivalently: Is $\mathbb{A}^n_{\mathbb{Q}}/G$ rational for every finite group $G$?*

As we know today, this is not the case already for almost all prime cyclic groups (Swan 1969, ..., Plans 2017).
Many other applications of HIT, e.g. Néron specialization.

# History Part 2: End of the 20th century (I)

**Question**

*For which other fields does HIT hold?*

**Definition (Lang)**

$K$ is **Hilbertian** if for $f \in K[t,x] \setminus K[x]$ irreducible there exist infinitely many $\tau \in K$ with $f(\tau, x)$ irreducible.

**Example**

Global fields are Hilbertian, local fields are not.

Applications of HIT to Galois theory work over arbitrary Hilbertian fields. For example:

**Theorem (Roquette 1975, Fried–Völklein 1992, Pop 1996)**

*A countable PAC field $K$ has $G_K \cong \hat{F}_\omega$ iff $K$ is Hilbertian.*

# History Part 2: End of the 20th century (I)

Many examples of Hilbertian fields come in the form of preservation theorems:

**Example**

If $K$ is Hilbertian, $L$ is Hilbertian in each of the following cases:

1. $L/K$ finite
2. $L/K$ small (i.e. algebraic and only finitely many finite subextensions of each degree)
3. $L/K$ abelian (Kuyk 1970)
4. $L/M$ proper finite, $M/K$ Galois (Weissauer 1982)
5. $K \subseteq L \subseteq K(A_{\mathrm{tor}})$, $A/K$ abelian variety (Jarden 2010, ...)

# History Part 2: End of the 20th century (II)

**Question**

*Can one specialize $\mathbb{A}^n_{\mathbb{Q}} \to \mathbb{A}^n_{\mathbb{Q}}/G$ even if $\mathbb{A}^n_{\mathbb{Q}}/G$ is not rational?*

For example, it would suffice that $\mathbb{A}^n_{\mathbb{Q}}/G$ is *stably rational*.

Let $X$ be a $K$-variety (in particular irreducible) with $\dim(X) \geq 1$.
A **cover** is a finite surjective morphism $Y \to X$ of $K$-varieties.

**Definition (Colliot-Thélène–Sansuc 1987, Serre 1991)**

Let $T \subseteq X(K)$. The set $T$ is of type

$(C_1)$ if $T \subseteq C(K)$, with $C \subsetneqq X$ closed,

$(C_2)$ if $T \subseteq \pi(Y(K))$, with $\pi\colon Y \to X$ a nontrivial cover.

It is **thin** if it is contained in a finite union of such sets.
The variety $X$ has **HP** if $X(K)$ is not thin.

HP is a birational property. Over $K = \mathbb{C}$, no variety has HP.
No elliptic curve $X = E$ over a number field $K$ has HP:
$E(K)/2E(K)$ is finite, and every coset of $2E(K)$ is of type $(C_2)$.

# History Part 2: End of the 20th century (II)

**Proposition**

*A $K$-variety $X$ has HP iff for every cover $\pi\colon Z \to X$ there exists a Zariski-dense set of $x \in X(K)$ with $\pi^{-1}(x)$ integral.*

**Proof.**

$\Longrightarrow$: Without loss of generality, $Z \to X$ is Galois and étale.
Let $\pi_i\colon Y_i \to X$, $i = 1, \dots, n$, be the nontrivial subcovers of $Z \to X$.
For every $x \in X(K)$ outside the thin set $\bigcup_{i=1}^{n} \pi_i(Y_i(K))$, $\pi^{-1}(x)$ is integral.

$\Longleftarrow$: Given $\pi_i\colon Y_i \to X$, $i = 1, \dots, n$, choose any Galois cover $Z \to X$ that dominates every $Y_i$. If $\pi^{-1}(x)$ is integral, then in particular $x \notin \pi_i(Y_i(K))$ for every $i$. □

So: $K$ is Hilbertian $\Leftrightarrow$ $\mathbb{A}^1_K$ has HP $\Leftrightarrow$ $\mathbb{P}^1_K$ has HP $\Leftrightarrow$ $\mathbb{P}^n_K$ has HP
In fact: $K$ is Hilbertian $\Leftrightarrow$ some $K$-variety $X$ has HP

# History Part 2: End of the 20th century (II)

**Proposition (Serre)**

*If $X$ has HP and $L/K$ is finite, then $X_L$ has HP.*

**Proposition (Colliot-Thélène–Sansuc 1987)**

*Let $f: Y \to X$ dominant with generic fiber geometrically irreducible. If $Y$ has HP, then $X$ has HP.*

They used this to show that for $K$ Hilbertian, any reductive linear algebraic group over $K$ has HP.

**Theorem (Ekedahl 1987, Colliot-Thélène 1988)**

*If $K$ is a number field and $X$ has WWA, then $X$ has HP.*

As linear algebraic groups over number fields have WWA (Sansuc), they have HP. Also by the Theorem, the following would solve IGP:

**Conjecture (Colliot-Thélène)**

Every smooth unirational variety over a number field $K$ has WWA.

# History Part 3: The last 10 years

**Question (Serre)**

Let $X, Y$ be $K$-varieties with HP. Does $X \times Y$ have HP?

BFP 2014: YES. Also proven by Wittenberg, van den Dries.
BFP deduces this from a fibration theorem, final form Luger 2022.

**Corollary (Bary-Soroker–F.–Petersen 2014)**

*If $K$ is Hilbertian, every linear algebraic group over $K$ has HP.*

Since then, more than 20 papers by more than 20 authors directly dealing with HP have appeared:

1. general results about HP and its preservation properties
2. HP for specific (families of) varieties (Corvaja, Zannier, Demeio, Gvirtz-Chen, ...)
3. generalizations of HP (Coccia, Luger, Streeter, ...)

# History Part 3: The last 10 years

**Theorem (Demeio 2018)**

*$\mathbb{A}^n_K/G$ has HP iff $K$ is Hilbertian and there is a linearly disjoint family $(L_i)_{i\in\mathbb{N}}$ of Galois extensions of $K$ with $\mathrm{Gal}(L_i/K) \cong G$.*

**Corollary**

*$K$ number field, $G$ solvable $\Longrightarrow \mathbb{A}^n_K/G$ has HP*

From now on, all $K$-varieties are assumed normal.

**Theorem (Corvaja–Zannier 2017)**

*If $K$ is a number field and $X$ is projective and has HP, then $X$ is simply connected (i.e. every nontrivial cover is ramified).*

# History Part 3: The last 10 years

**Definition (Corvaja–Zannier 2017)**

A set $T \subseteq X(K)$ is **strongly thin** if it is contained in finitely many sets of type $(C_1)$ and $\pi(Y(K))$ with $\pi\colon Y \to X$ a ramified cover.
$X$ has **WHP** if $X(K)$ is not strongly thin.

$\mathbb{P}^1_K$ has WHP $\Leftrightarrow$ $\mathbb{P}^1_K$ has HP $\Leftrightarrow$ $K$ is Hilbertian

**Example**

If $K$ is a number field, $E$ an elliptic curve over $K$ with $E(K)$ Zariski-dense, then $E$ has WHP: Riemann–Hurwitz + Faltings.

**Question (Corvaja–Zannier 2017)**

If $K$ is a number field, does every $K$-variety $X$ with $X(K)$ Zariski-dense have WHP?

WHP implies a statement on irreducible fibers (but not conversely).
So if YES, this would in particular imply a solution to IGP.

# Permanence

**Theorem (Javanpeykar–Wittenberg, CDJLZ 2022)**

*If $K$ is finitely generated and $X, Y$ are smooth proper $K$-varieties with WHP, then $X \times Y$ has WHP.*

**Proposition (BFP 2024)**

*Let $X$ have HP/WHP. If $L/K$ is finitely generated or small, then $X_L$ has HP/WHP.*

**Proposition (CDJLZ 2022, BFP 2024)**

*Let $f : Y \to X$ smooth surjective. Assume $f$ proper or with geometrically irreducible generic fiber. If $Y$ has WHP, then so does $X$.*

It follows that if some $K$-variety has WHP, then $K$ is Hilbertian.

**Example (BFP 2024)**

No variety over $\mathbb{Q}^{\mathrm{tr}}$ has WHP. Every variety over $\mathbb{Q}^{\mathrm{tr}}(i)$ has HP.

# Abelian varieties

**Theorem (Corvaja–Demeio–Javanpeykar–Lombardo–Zannier 2022)**

*If $K$ is finitely generated and $A$ an abelian variety over $K$ with $A(K)$ Zariski-dense, then $A$ has WHP.*

By the above this implies the same result over any small extension $K$ of a finitely generated field. What about $K = \mathbb{Q}^{\mathrm{ab}}$ or $K = \mathbb{Q}(A_{\mathrm{tor}})$?

**Example**

Let $X$ be a $\mathbb{Q}$-variety with $X(\mathbb{Q}) = X(\mathbb{Q}^{\mathrm{ab}})$.
Then $X_{\mathbb{Q}^{\mathrm{ab}}}$ does not have WHP:
If $\pi\colon Y \to X$ is any abelian cover, $X(\mathbb{Q}^{\mathrm{ab}}) = X(\mathbb{Q}) \subseteq \pi(Y(\mathbb{Q}^{\mathrm{ab}}))$.

**Conjecture (Frey–Jarden 1974)**

Every abelian variety $A$ over $\mathbb{Q}$ has infinite rank over $\mathbb{Q}^{\mathrm{ab}}$.

Known for elliptic curves and many Jacobians (Rosen–Wong 2002, Petersen 2006, Im–Larsen 2013, ...) and would hold if $\mathbb{Q}^{\mathrm{ab}}$ was *large* (F.–Petersen 2010).

# Abelian varieties

**Theorem (BFP 2023)**

*Let $K$ be a number field. If the Frey–Jarden conjecture holds, then every abelian variety over $K^{\mathrm{ab}}$ has WHP.*

**Corollary**

*If $E$ is an elliptic curve over $\mathbb{Q}$, then $E_{\mathbb{Q}^{\mathrm{ab}}}$ has WHP.*

**Theorem (Gajda–Petersen 2024)**

*Let $K$ be a number field and $A, B$ abelian varieties over $K$. If the Frey–Jarden conjecture holds, then $A_{K(B_{\mathrm{tor}})}$ has WHP.*

**Corollary (BFP 2023)**

*If $E$ is an elliptic curve over $\mathbb{Q}$ and $B$ an abelian variety over $\mathbb{Q}$, then $E_{\mathbb{Q}(B_{\mathrm{tor}})}$ has WHP.*

Gajda–Petersen extend a recent result by Checcoli–Dill on $\mathrm{Gal}(K(B_{\mathrm{tor}})/K)$ and combine it with BFP23 and BFP24.

# About the proof

What is really proven:

**Theorem (Corvaja–Demeio–Javanpeykar–Lombardo–Zannier 2022)**

*Let $K$ finitely generated, $A$ an abelian variety over $K$, $\Omega \leq A(K)$ a Zariski-dense subgroup, and $\pi_i\colon Y_i \to A$ a cover with no nontrivial étale subcovers ($i = 1, \ldots, r$). Then there exists a finite index coset $C \subseteq \Omega$ such that each $\pi_i^{-1}(c)$ is integral for every $c \in C$.*

**Theorem (BFP 2023)**

*Let $K$ be finitely generated, $A$ an abelian variety over $K$, and $L/K$ an abelian extension such that $\dim_{\mathbb{Q}}(A_0(L) \otimes_{\mathbb{Z}} \mathbb{Q}) = \infty$ for every nonzero homomorphic image $A_0$ of $A_L$.*
*Then $A_L$ has WHP.*

# About the proof

Modulo technical details, it (roughly) suffices to show that
if $\pi\colon Y \to A$ is a Galois cover with Galois group $\Gamma$ and
no nontrivial étale subcover there exists $x \in A(L)$ with $\pi_L^{-1}(x)$ integral.

**Idea** (following Haran 1999): Choose nontrivial finite Galois subextension $M/K$ of $L/K$ with Galois group $G$, and consider

$$\mathrm{res}_{M/K}(\pi_M)\colon \mathrm{res}_{M/K}(Y_M) \to \mathrm{res}_{M/K}(A_M) =: B.$$

Then

$$\rho\colon \mathrm{res}_{M/K}(Y_M)_M \cong \prod_{\sigma\in G} Y_M^\sigma \to \prod_{\sigma\in G} A_M^\sigma \cong B_M \to B$$

is a Galois cover with Galois group $\Gamma \wr G = \Gamma^G \rtimes G$ (very non-abelian!) and no nontrivial geometrically integral étale subcovers.
However, $B$ need not have WHP!
Restricting $\rho$ to the diagonal $\Delta\colon A \to B$ gives a "cover" $Z \to A$ which however is reducible.

**Solution**: First translate $\pi_M\colon Y_M \to A_M$ by suitable $t \in A(M)$ to make sure that some irreducible component of $Z$ has Galois group a sufficiently big subgroup of $\Gamma \wr G$, by controlling ramification.

# About the proof

## Lemma

*Let $A$ be an abelian variety over $K$, and let $L/K$ be an abelian extension. Let $\alpha\colon \prod_{i=1}^k A_i \to A$ be an isogeny where each $A_i$ is a simple abelian variety over $K$ with $\mathrm{rk}(A_i(L)/A_i(K_1)) = \infty$ for every finite subextension $K_1/K$ of $L/K$.*
*For every finite set $S \subseteq A(\bar{K})$ and proper abelian subvarieties $B_1, \dots, B_r \subsetneq A$ there exist $t \in A(L)$ and $\sigma \in \mathrm{Gal}(L/K)$ with*

$$t - \sigma(t) \notin S + \bigcup_{j=1}^{r} B_j(\bar{K}).$$

## Lemma

*Let $\Gamma$ and $G$ be groups and let $H \leq \Gamma \wr G$ such that $\mathrm{pr}(H) = G$ and $H \cap \Gamma^G$ surjects onto $\Gamma^{\{1,\sigma\}}$ for some $1 \neq \sigma \in G$ under the restriction map $f \mapsto f|_{\{1,\sigma\}}$. Then for every $H' \leq N \leq \Gamma \wr G$ we have that $N \cap \Gamma^G$ surjects onto $\Gamma^{\{1\}}$ under $f \mapsto f|_{\{1\}}$.*

# References


1. L. Bary-Soroker, A. Fehm and S. Petersen. On varieties of Hilbert type. *Ann. Inst. Fourier*, 2014.
2. L. Bary-Soroker, A. Fehm and S. Petersen. Ramified covers of abelian varieties over torsion fields. *J. angew. reine Math.*, 2023.
3. L. Bary-Soroker, A. Fehm and S. Petersen. Hilbert properties under base change in small extensions. arXiv:2312.16219 [math.NT]
4. S. Checcoli and G. Dill. On a Galois property of fields generated by the torsion of an abelian variety. arXiv:2306.12138 [math.NT]
5. J.-L. Colliot-Thélène and J.-J. Sansuc. Principal homogeneous spaces under flasque tori: applications. *J. Algebra*, 1987.
6. P. Corvaja, J. L. Demeio, A. Javanpeykar, D. Lombardo and U. Zannier. On the distribution of rational points on ramified covers of abelian varieties. *Compositio Math.*, 2022.
7. P. Corvaja and U. Zannier. On the Hilbert property and the fundamental group of algebraic varieties. *Math. Z.*, 2017.
8. J. L. Demeio. Non-rational varieties with the Hilbert Property. *Internat. J. Number Theory*, 2020.
9. W. Gajda and S. Petersen. Finiteness properties of torsion fields of abelian varieties. arXiv:2401.05805 [math.NT]
10. D. Haran. Hilbertian fields under separable algebraic extensions. *Invent. Math.*, 1999.
11. C. Luger. The Hilbert property for arithmetic schemes. *Acta Arithmetica*, 2022.