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Étale homotopy theory and applications

Brieuc Lair (Bordeaux, FR), Drimik Roy-Chowdhury (IMJ-PRG, FR), and Naganori Yamaguchi (Science Tokyo, JP) [2]

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The aim of this note is as follows: first, to introduce the main constructions of the étale topological type $X_{\text{ét}}$ via (rigid) étale hypercoverings, together with its basic invariants (étale homotopy groups and cohomology with local coefficients), following [Fri82]; second, to explain how these topological types provide one of a framework for anabelian geometry, in particular the result of Schmidt–Stix [SS16], illustrated by concrete examples[3].

[3] The program and all the information about the *Atelier* can be found on the AHGT website.

Contents

Étale homotopy theory, built on the work of M. Artin and B. Mazur in the 1960s [AM69], is an analogue in algebraic geometry of classical homotopy theory for topological spaces. Roughly speaking, for any scheme X, they consider the category $\mathrm{HR}(X)$ of all (étale) hypercoverings of X and construct a certain Verdier functor $\mathrm{HR}(X) \to \mathbf{Set}^{\Delta^{\mathrm{op}}}$.

By passing to the homotopy category $\mathrm{Ho}(\mathrm{HR}(X))$ of $\mathrm{HR}(X)$, one obtains a pro-simplicial set of X, which is called *the étale homotopy type* of X.

$$X_{\mathrm{ht}} \colon \mathrm{Ho}(\mathrm{HR}(X)) \to \mathbf{Set}^{\Delta^{\mathrm{op}}}$$

A well-known technical issue of this construction is that $\mathrm{HR}(X)$ need not be cofiltered, so one must pass to a homotopy category in the construction of the resulting pro-object.

This issue was resolved by E. M. Friedlander [Fri82] by introducing the category $\mathrm{HRR}(X)$ of *rigid hypercoverings* and the *étale topological type* $X_{\text{ét}}$ of X, which is a pro-simplicial set (i.e. a pro-object of simplicial sets). It recovers the étale cohomology of locally constant sheaves and Grothendieck's étale fundamental group of X[4].

By relying on previous anabelian results of [Tam97] and [Moc99], the étale topological type allows a reformulation [5] of Grothendieck's classical anabelian conjectures[6] for hyperbolic curves:

> *Let k be a finitely generated field over $\mathbb{Q}$, and let X, Y be hyperbolic curves over k. Then the natural map*
>
> $$\mathrm{Isom}_k(X,Y) \longrightarrow \mathrm{Isom}_{\mathrm{Ho}(\mathrm{Pro}\text{-}\mathbf{sSet})\downarrow k_{\text{ét}}}(X_{\text{ét}}, Y_{\text{ét}}),$$
>
> *is bijective. In other words, X and Y are k-isomorphic if and only if their étale topological types are isomorphic over $k_{\text{ét}}$ in the homotopy category of pro-simplicial sets.*

This approach leads to a similar higher-dimensional anabelian statement, (a) when X and Y are strongly hyperbolic Artin neighborhoods, and (b) the *existence of an anabelian Zariski-open basis for every smooth variety*[7].

[4] Modern work, which is not treated in these notes, continues to expand the available toolkit and comparison theorems; see for instance [HHW24] and [Mef25].

[5] Alexander Schmidt and Jakob Stix. Anabelian geometry with étale homotopy types. *Ann. of Math. (2)*, 184(3):817–868, 2016

[6] Proved in particular by H. Nakamura – for genus 0 curves, see [Nak90], A. Tamagawa – for affine curves, see [Tam97], and S. Mochizuki – for general case, see [Moc99]; see also the survey [NTM01]

[7] See also Y. Hoshi [Hos20] for a more general result by different techniques.

Simplicial schemes and their cohomology

Motivation from topology

The goal is to understand the *homotopy type* of an object. For X a *CW complex*, an open covering $X = \bigcup_i U_i$ by open sets such that any finite intersection of the U_i is contractible or empty is called *a good covering* of X. We can represent this via a topological quotient $\mathcal{U} = \bigsqcup U_i \to X$, and this covering has the property that

$$|\pi_0(\mathrm{Nerve}(\mathcal{U} \to X))| \xrightarrow{\cong} X$$

up to homotopy equivalence where $|\pi_0(\mathrm{Nerve}(\mathcal{U} \to X))|$ denotes the *geometric realization* of the simplicial set $\pi_0(\mathrm{Nerve}(\mathcal{U} \to X))$. That is, we can recover the homotopy type of such CW complexes via the simplicial set $\mathrm{Nerve}(\mathcal{U} \to X)$.

This idea can be massively generalized and applied to a wider class of geometric objects, namely locally contractible topological spaces, by using hypercoverings[8]. Let us illustrate the case of a topological manifold that admits an open covering by contractible connected open subsets U_i. One covers again the intersections $U_{i,j} = U_i \times_X U_j$ for each i, j and repeats this process for n-fold intersections $U_{i_1,\dots,i_n}$ to build a *hypercovering*[9] $U_\bullet \to X$. Once again, one obtains an isomorphism

$$|\pi_0(\mathcal{U}_\bullet)| \xrightarrow{\cong} X$$

up to homotopy equivalence[10]. A consequence of this identification is that *any topological manifold has the homotopy type of a CW complex*. Moreover, the singular cohomology groups of X are detected by the covering $U_\bullet$ via the Čech cohomology groups along the hypercovering: for any sheaf of abelian groups $\mathcal{F}$ on X, we have a canonical isomorphism

$$R\Gamma(X, \mathcal{F}) \xrightarrow{\cong} R\Gamma_{\text{Čech}}(U_\bullet, \mathcal{F}|_{U_\bullet}) \qquad (\star)$$

The notion of a hypercovering is a refinement of Čech nerves by allowing more freedom at each level in the cosimplicial covering.

We have seen that the notion of hypercoverings in topology captures the theory of singular cohomology groups and the underlying homotopy type of a space for topological manifolds and CW complexes. The goal of étale homotopy theory is to realize this in algebro-geometric contexts. We will see that hypercoverings recover étale sheaf cohomology completely for (simplicial) schemes and that the étale homotopy type of a scheme recovers at least the cohomology of locally constant étale sheaves.

[8] This is the approach developed by Artin and Mazur.

[9] That is, we cover the manifold at each step n by contractible open connected sets.

[10] Actually, the use of hypercoverings is not always strictly necessary: for example, one can use only the Čech nerves to recover the homotopy type of any paracompact topological space.

Étale site of a simplicial scheme

Let $X_\bullet \in \mathbf{sSch}$ be a simplicial scheme [11,12]. We let $\text{Ét}(X_\bullet)$ denote the category whose objects are étale maps $U \to X_n$ for some $n \geq 0$ with arrows given by commutative squares

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X_n & \longrightarrow & X_m \end{array}$$

where the bottom arrow arises in the simplicial scheme $X_\bullet$ (that is[13], the datum of an arrow in $\text{Ét}(X_\bullet)$ necessarily includes the choice of an arrow in Δ). An étale covering for an object $U \to X_n$ in $\text{Ét}(X_\bullet)$ is a usual étale covering of U over X_n. With this, we see that $\text{Ét}(X_\bullet)$ is a site.

We note that while $U \to X_n$ may not arise from a map of simplicial schemes, there exists a functorial way to extend $U \to X_n$ to a map of simplicial schemes $U_\bullet \to X_\bullet$ that:

[11] Eric M. Friedlander. *Étale homotopy of simplicial schemes*, volume No. 104 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1982

[12] For any category $\mathcal{C}$, denote by $s\mathcal{C}$ the *category of simplicial objects* in $\mathcal{C}$, i.e. $\mathbf{Func}(\Delta^{\text{op}}, \mathcal{C})$.

[13] See [Fri82, Definition 1.4].

Proposition 1 ([Fri82, Proposition 1.5]). *Let $X_\bullet$ be a simplicial scheme. For any $n \geq 0$, the functor $(\mathbf{sSch})_{X_\bullet} \to \mathbf{Sch}_{X_n}$ by recording the map on*

n-simplices has a right adjoint $\Gamma_n^{X_\bullet}(-)$. Moreover, if $Z \to X_n$ is étale (resp., surjective), then $\Gamma_n^{X_\bullet}(Z) \to X_\bullet$ is étale (resp., surjective) in each degree. Moreover, the map $\Gamma_n^{X_\bullet}(Z)_n \to X_n$ factors through $Z \to X_n$.

Thus, any étale surjective map $Z \to X_n$ is dominated by the restriction to n-simplices of an étale surjective map $U_\bullet \to X_\bullet$ of simplicial schemes.

Sheaves and cohomology

Definition 2. An *(étale) presheaf of sets* on $X_\bullet$ is a functor $\text{Ét}(X_\bullet)^{\text{op}} \to$ **Set**. We denote $\text{Psh}(X_\bullet) = \text{Psh}(X_\bullet; \mathbf{Set})$ as the category of set-valued presheaves on $X_\bullet$.

For notational convenience, any presheaf $\mathcal{F} \in \text{Psh}(X_\bullet)$ evaluated on $U \to X_\bullet$ (i.e., explicitly, an étale morphism $U \to X_n$ for some n) is denoted $\mathcal{F}(U)$.

Definition 3 ([Fri82, Definition 2.1]). A *sheaf of sets* on $X_\bullet$ is a presheaf of sets satisfying the sheaf condition with respect to the Grothendieck topology on $\text{Ét}(X_\bullet)$ defined above. We denote $\text{Shv}_{\text{ét}}(X_\bullet) = \text{Shv}_{\text{ét}}(X_\bullet; \mathbf{Set})$ as the category of set-valued sheaves on $X_\bullet$.

Explicitly, the datum of a sheaf $\mathcal{F}$ on $\text{Ét}(X_\bullet)$ is exactly the data of sheaves $\mathcal{F}_n$ on the small étale site of X_n for each $n \geq 0$ with a morphism $\mathcal{F}_m \to \alpha_* \mathcal{F}_n$ in $\text{Ét}(X_m)$ for each $\alpha : X_m \to X_n$ arrow in the simplicial scheme $X_\bullet$. We moreover demand the compatibility condition that $\alpha^*(\beta^* \mathcal{F}_d) \to \alpha^* \mathcal{F}_m \to \mathcal{F}_n$ is equal to the map $(\beta \circ \alpha)^* \mathcal{F}_d \to \mathcal{F}_m$ associated to the map $\beta \circ \alpha : [d] \to [m] \to [n]$.

Example 4. Two key examples of simplicial sheaves.

1. If $X_\bullet$ is the constant simplicial scheme associated to X_0, then a sheaf on $\text{Ét}(X_\bullet)$ is equivalent to a cosimplicial object of sheaves on the usual small étale site $\text{Ét}(X_0)$ of X_0.

2. For any scheme Y, we have a sheaf h_Y on $\text{Ét}(X_\bullet)$ by sending $U \to X_n$ to $\text{Hom}_{\mathbf{Sch}}(U, Y)$. Any $V \to X_m \in \text{Ét}(X_\bullet)$ also determines a sheaf by sending $U \to X_n$ to $\text{Hom}_{\text{Ét}(X_\bullet)}(U, V)$. These are the representable sheaves.

Definition 5. A *geometric point* of a simplicial scheme $X_\bullet$ is a morphism $\underline{x} : \text{Spec}(\Omega) \to X_n$ for some $n \geq 0$ where Ω is a separably closed field. The *stalk* of a sheaf of abelian groups $\mathcal{F}$ on $\text{Ét}(X_\bullet)$ at a point $\underline{x}$ is $\underline{x}^* \mathcal{F}_n(\text{Spec}(\Omega))$.

Proposition 6 ([Fri82, Proposition 2.2]). *If $X_\bullet$ is a simplicial scheme, then* $\text{Shv}_{\text{ét}}(X_\bullet; \mathbf{Ab})$ *is a Grothendieck abelian category. Moreover, a sequence of sheaves is exact if and only if the induced sequence of stalks for any geometric point is exact.*

Thus, we are now able to define étale cohomology theory on any simplicial scheme.

Definition 7 ([Fri82, Definition 2.3])**.** Let $X_\bullet$ be any simplicial scheme. We have $R\Gamma(X_\bullet, -)$ defined as the right derived functor sending an abelian sheaf on $\text{Ét}(X_\bullet)$ to the equalizer of

$$F(X_0) \underset{d_1^*}{\overset{d_0^*}{\rightrightarrows}} F(X_1).$$

We denote its cohomology groups $H^i(X_\bullet, -)$ as $H^i_{\text{ét}}(X_\bullet, -)$[14].

[14] Equivalently, one can show that $H^i(X_\bullet, -) = \text{Ext}^i_{\text{Shv}_{\text{ét}}(X_\bullet;\mathbf{Ab})}(\underline{\mathbf{Z}}, -)$.

A natural question is how the cohomology groups on $\text{Ét}(X_\bullet)$ are related to the cohomology groups along n-simplices $\text{Ét}(X_n)$ as we vary n: the former is built out of the latter via a convergent spectral sequence.

Proposition 8 ([Fri82, Proposition 2.4])**.** *Let $X_\bullet$ be a simplicial scheme, and let $\mathcal{F} \in \text{Shv}_{\text{ét}}(X_\bullet; \mathbf{Ab})$. Then there exists a first quadrant spectral sequence*

$$E_2^{p,q} = H^p_{\text{ét}}(X_q, \mathcal{F}_q) \Longrightarrow H^{p+q}_{\text{ét}}(X_\bullet, \mathcal{F}).$$

Corollary 9. *If $\mathcal{F} \in \text{Shv}_{\text{ét}}(X_\bullet; \mathbf{Ab})$ is such that each $\mathcal{F}_n \in \text{Shv}_{\text{ét}}(X_n; \mathbf{Ab})$ is injective, then $\mathcal{F}$ is acyclic if and only if $\mathcal{F}(X_n) = 0$ for all $n > 0$.*

Proof. By definition, a first quadrant convergent spectral sequence $E_2^{p,q} \to H^{p+q}$ implies a filtration

$$H^n = F^0 H^n \supseteq F^1 H^n \supseteq \cdots \supseteq F^n H^n = \{0\}$$

such that $F^p H^{p+q} / F^{p+1} H^{p+q} \cong E_\infty^{p,q}$. Since $\mathcal{F}_n$ are all acyclic, we conclude that

$$H^0(X_q, \mathcal{F}_q) \cong E_\infty^{0,q} = F^0 H^q(X_\bullet, \mathcal{F}) / F^1 H^q(X_\bullet, \mathcal{F})$$

for any $q \geq 0$. □

Bisimplicial schemes We next generalize the above results to bisimplicial schemes.

Definition 10. A *bisimplicial scheme* is a functor $\Delta^{\text{op}} \times \Delta^{\text{op}} \to \mathbf{Sch}$. We denote such a functor as $X_{\bullet,\bullet}$.

Exactly analogous to the case of simplicial schemes, we can define the étale site $\text{Ét}(X_{\bullet,\bullet})$ and étale topos $\text{Ét}(X_{\bullet,\bullet})\tilde{} = \text{Shv}_{\text{ét}}(X_{\bullet,\bullet}; \mathbf{Ab})$ associated to a bisimplicial scheme. We define

$$H^i(X_{\bullet,\bullet}, \mathcal{F}) := \text{Ext}^i_{\text{Shv}_{\text{ét}}(X_{\bullet,\bullet};\mathbf{Ab})}(\mathbf{Z}, \mathcal{F})$$

where $\mathbf{Z}$ is the constant abelian sheaf with fiber $\mathbf{Z}$.

Definition 11. For $X_{\bullet,\bullet}$ a bisimplicial scheme, we let $\Delta(X_{\bullet,\bullet})$ be the simplicial scheme $\Delta \ni n \mapsto X_{n,n}$.

For the purposes of cohomology, we can replace a bisimplicial scheme by its diagonal simplicial scheme by the following proposition.

Proposition 12 ([Fri82, Proposition 2.5]). *Let $X_{\bullet,\bullet}$ be a bisimplicial scheme. Then there is an isomorphism of δ-functors:*

$$H^*(X_{\bullet,\bullet}, -) \longrightarrow H^*(\Delta(X_{\bullet,\bullet}), (-)^\Delta)$$

on $\mathrm{Shv}_{\text{ét}}(X_{\bullet,\bullet}; \mathbf{Ab})$ *where* $(-)^\Delta$ *is the restriction functor.*

The following is the bisimplicial analogue of proposition 8.

Proposition 13 ([Fri82, Proposition 2.6]). *Let $X_{\bullet,\bullet}$ be a bisimplicial scheme and $\mathcal{F}$ be an abelian sheaf on $X_{\bullet,\bullet}$. Then there exists a first quadrant convergent spectral sequence*

$$E_2^{p,q} = H^p(X_{q,\bullet}, \mathcal{F}_{q,\bullet}) \Longrightarrow H^{p+q}(X_{\bullet,\bullet}, \mathcal{F})$$

natural in $X_{\bullet,\bullet}$ and $\mathcal{F}$.

The étale topological type of a simplicial scheme

Cohomology via hypercoverings

There are several issues that arise in transporting to algebro-geometric objects the idea from CW complexes and locally contractible spaces that homotopy types and cohomology groups can be detected by coverings: (1) which topology captures the "true" homotopy type of a scheme, (2) how to capture the notion of hypercoverings in an arbitrary topos, and (3) whether any single hypercovering captures the homotopy type of a scheme.

For point (3), the analogue of the fundamental group is that the étale fundamental group is the automorphism group of a fiber functor and this fiber functor is pro-representable, but not representable: one has to use the entire system of coverings as a functor, instead of a single étale covering of a scheme to compute the étale fundamental group.

Note that in the case of cohomology already, one shows that Čech nerves, and more generally, any single hypercovering is insufficient[15] to recover sheaf cohomology as in ($\star$). On the other hand, following a result of Verdier, it turns out that if we consider the colimit over all hypercoverings, then we recover étale cohomology:

[15] For the failure of Čech cohomology to detect acyclicity, we refer to [Fri82, p. 23]; for a positive result under quasi-projective and noetherian assumptions, see Corollary 3.9 ibid. or Corollary 23.

Proposition 14 ([AM69, Theorem 8.16]). *Let $\mathcal{F} \in \mathrm{Shv}_{\text{ét}}(X_\bullet; \mathbf{Ab})$. Then the natural map*

$$R\Gamma(X, \mathcal{F}) \xrightarrow{\cong} \operatorname*{colim}_{\text{hypercoverings } \mathcal{U}_\bullet \to X} R\Gamma_{\check{C}ech}(\mathcal{U}_\bullet, \mathcal{F})$$

is an isomorphism in $D(X, \mathbf{Z})$.

Definition 15. We let $\mathrm{Pro}(\mathcal{C})$ denote the category of functors $I^{\mathrm{op}} \to \mathcal{C}$ where I is a cofiltered category[16].

[16] For a connected scheme X, the cofiltered category of connected finite étale coverings of X computes the fundamental group of X (with respect to a étale point of X). That is, the universal covering of X is really an object that lives in $\mathrm{Pro}(\mathbf{Sch}_X)$.

The goal of this section is to prove Proposition 14 for simplicial schemes. We do this gradually by first dealing with Čech nerves and Čech cohomology.

Čech cohomology Let $X_\bullet$ be a simplicial scheme. An *étale covering* $U_\bullet \to X_\bullet$ of $X_\bullet$ is a map of simplicial schemes such that the fiber over $n \in \Delta$ is an étale surjective map.

The Čech nerve of $U_\bullet \to X_\bullet$ is the bisimplicial scheme $N_{X_\bullet}(U_\bullet)$ defined by

$$N_{X_\bullet}(U_\bullet)_{p,q} = (N_{X_p}(U_p))_q$$

the $(q+1)$-fold fiber product of U_p with itself over X_p. For any abelian presheaf $\mathcal{P} \in \mathrm{Psh}(X_\bullet; \mathbf{Ab})$, we define $\mathcal{P}(N_{X_\bullet}(U_\bullet))$ as the bicochain complex given in bicodimension (p,q) as $\mathcal{P}(N_{X_\bullet}(U_\bullet)_{p,q})$ with differentials as the alternating sum over the bicomplex.

Definition 16 (Čech cohomology)**.** Let $X_\bullet$ be a simplicial scheme and $\mathcal{P}$ be an abelian sheaf on $X_\bullet$. For any $i \geq 0$, define the *Čech cohomology* of $X_\bullet$ with values in $\mathcal{P}$ in degree i by

$$\check{H}^i(X_\bullet, \mathcal{P}) = \operatorname*{colim}_{U_\bullet \to X_\bullet} H^i(\mathcal{P}(N_{X_\bullet}(U_\bullet)_{p,q}))$$

where the colimit is the category of Čech nerves of étale coverings $U_\bullet \to X_\bullet$ of simplicial schemes. This is a colimit over a cofiltered category.

As always, $\check{H}^*(X_\bullet, -)$ is a δ-functor on the category of abelian presheaves[17] on $X_\bullet$. The following is the Čech cohomology analogue of proposition 8.

[17] See [Fri82, Proposition 3.1].

Proposition 17 ([Fri82, Proposition 3.2])**.** *Let $X_\bullet$ be a simplicial scheme and $\mathcal{P}$ be an abelian presheaf on $X_\bullet$. Then there exists a first quadrant spectral sequence*

$$E_2^{p,q} = \check{H}^p(X_q, \mathcal{P}_q) \Longrightarrow \check{H}^{p+q}(X_\bullet, \mathcal{P})$$

where $\mathcal{P}_q$ is the restriction of $\mathcal{P}$ to the étale site X_q.

At this point, for a simplicial scheme, we have the abstract étale cohomology groups $H^*_{\text{ét}}$ and the "more" explicit Čech cohomology groups $\check{H}^*$. Moreover, each notion of cohomology is related to the cohomology of its simplices via a first quadrant convergent spectral sequence (Propositions 8 and 17). Proposition 21 and Theorem 22 below together show that there exists a unique map of δ-functors $\check{H}^* \to H^*_{\text{ét}}$ for any simplicial scheme. However, this morphism is not an isomorphism in general[18]. The problem is that Čech nerves are not a fine enough operation.

[18] An important case of when it is so is Proposition 23 below.

Formalism of hypercoverings in a topos For a simplicial set $X_\bullet$ (or in general a simplicial object in a category), we let $\operatorname{sk}_n X_\bullet$ denote the truncation of $X_\bullet$ at level n, and cosk_n is the right adjoint to sk_n:

$$s\mathcal{T} \underset{\operatorname{cosk}_n}{\overset{\operatorname{sk}_n}{\rightleftarrows}} s\mathcal{T}_{\leq n}$$

Definition 18. Let τ be any topos. We say $X_\bullet \to Y$ is a hypercovering if $X_0 \to Y$ is a surjection and for all $n \geq 0$, each map $X_{n+1} \to (\operatorname{cosk}_n \operatorname{sk}_n X_\bullet)_{n+1}$ is surjective.

The description of sk_n and cosk_n for arbitrary n is a bit involved, but one can make it explicit in the cases $n = 0$. If $n = 0$, $\operatorname{sk}_0(X_\bullet) = X_0$ and for an object $A \in \tau$, the coskeleton $\operatorname{cosk}_0(A)$ is the constant simplicial set with value A. More generally, one has sk_n^S and cosk_n^S as an adjoint pair between simplicial S-schemes and n-truncated simplicial S-schemes for a base scheme S. Then cosk_0^S of an S-scheme X is exactly the Čech nerve along the structure map $X \to S$.

Example 19 (Čech nerves are hypercoverings.)**.** Let Y be an S-scheme and consider an étale covering $X \to Y$ over S. If we let $X_\bullet \to Y$ be the Čech nerve associated to $X \to Y$, i.e., X_n is the $(n+1)$-fold fiber product of X over Y, then one can see that $X_{n+1} = (\operatorname{cosk}_n^S \operatorname{sk}_n^S X_\bullet)_{n+1}$:

$$X_1 = (\operatorname{cosk}_0^S \operatorname{sk}_0^S X_\bullet)_1 = X \times_Y X \text{ as remarked above}$$
$$X_2 = (\operatorname{cosk}_1^S \operatorname{sk}_1^S X_\bullet)_2 \cong (X_1 \times_{X_0} X_1) \times_{X_0 \times_S X_0} X_1 \cong X \times_Y X \times_Y X$$
$$\cdots\cdots$$

The importance of hypercoverings is that in general for schemes, if one has a hypercovering $f : X_\bullet \to Y$ and $\mathscr{F}$ is a Λ-module on Y for Λ an abstract ring, one has an equivalence of categories

$$R\Gamma(Y, \mathscr{F}) \xrightarrow{\cong} \varprojlim_{n \in \Delta} R\Gamma(X_n, f^*\mathscr{F})$$

$$\mathscr{D}(Y, \Lambda) \xrightarrow{\cong} \varprojlim_{n \in \Delta} \mathscr{D}(X_n, \Lambda)$$

where we consider the inverse limit of the derived ∞-categories, i.e., the sheaf condition is satisfied for hypercoverings on the level of the derived ∞-categories of Λ-modules.

Étale hypercoverings for simplicial schemes The following is a generalization of Čech nerves of simplicial schemes.

Definition 20 ([Fri82, Definition 3.3])**.** Let $X_\bullet$ be a simplicial scheme. An étale hypercovering $U_{\bullet,\bullet} \to X_\bullet$ is a bisimplicial scheme over $X_\bullet$ with the property that $U_{n,\bullet} \to X_n$ is a hypercovering for each $n \geq 0$ in the sense of definition 18.

One can show that the homotopy category[19] of Čech nerves of étale coverings $U_\bullet \to X_\bullet$ is a cofiltered category.

Similarly, the full subcategory of simplicial schemes that are hypercoverings of X denoted $\mathrm{Cov}_{\mathrm{hyp}}(X_\bullet)$ is not cofiltered, but its homotopy category is. Let $\mathrm{Ho}(\mathrm{Cov}_{\mathrm{hyp}}(X_\bullet))$ denote the homotopy category of $\mathrm{Cov}_{\mathrm{hyp}}(X_\bullet)$. There is a natural morphism

$$\left\{ \begin{matrix} \text{homotopy category of Čech nerves} \\ \text{of étale coverings of } X_\bullet \end{matrix} \right\} \longrightarrow \mathrm{Ho}(\mathrm{Cov}_{\mathrm{hyp}}(X_\bullet))$$

For emphasis, we record these results as a proposition.

[19] For a brief reminder on model categories, we refer to § *Note: Model Categories in Brief* at the end of this section.

Proposition 21. *The homotopy category* $\mathrm{Ho}(\mathrm{Cov}_{\mathrm{hyp}}(X_\bullet))$ *of étale hypercoverings of a simplicial scheme and the homotopy category of Čech nerves of étale coverings of* $X_\bullet$ *are both cofiltered categories.*

Thus, the colimit functor along $\mathrm{Ho}(\mathrm{Cov}_{\mathrm{hyp}}(X_\bullet))$-shaped diagrams (or for Čech nerves) in **Ab**

$$\begin{aligned} \mathbf{Func}(\mathrm{Ho}(\mathrm{Cov}_{\mathrm{hyp}}(X_\bullet))^{\mathrm{op}}, \mathbf{Ab}) &\longrightarrow \mathbf{Ab} \\ F &\longmapsto \operatorname{colim} F \end{aligned}$$

is an exact functor.

Theorem 22 ([Fri82, Theorem 3.8])**.** *Let* $X_\bullet$ *be a simplicial scheme. There is a natural isomorphism of* δ*-functors on* $\mathrm{Shv}_{\text{ét}}(X_\bullet; \mathbf{Ab})$

$$H^*(X_\bullet, -) \xrightarrow{\cong} \operatorname*{colim}_{U_{\bullet,\bullet} \to X_\bullet} H^*(U_{\bullet,\bullet}, -)$$

where the colimit is over the homotopy category $\mathrm{Ho}(\mathrm{Cov}_{\mathrm{hyp}}(X_\bullet))$ *of bisimplicial hypercoverings of* $X_\bullet$*.*

Using the previous theorem, one can generalize Artin's result [Art71, Corollary 4.2] that sheaf cohomology equals Čech cohomology for quasiprojective schemes over a noetherian ring.

Corollary 23 ([Fri82, Corollary 3.9])**.** *Let* $X_\bullet$ *be a simplicial scheme such that each* X_n *is quasiprojective over a noetherian ring for each* $n \geq 0$*. Then the natural map of* δ*-functors on* $\mathrm{Shv}_{\text{ét}}(X_\bullet; \mathbf{Ab})$

$$\check{H}^*(X_\bullet, -) \xrightarrow{\cong} H^*(X_\bullet, -)$$

is an isomorphism.

Étale homotopy and étale topological type

As seen in Theorem 22, the sheaf cohomology of a simplicial scheme $X_\bullet$ is determined by its hypercoverings. The étale homotopy type as an object of $\mathrm{Pro}(\mathrm{Ho}(\mathrm{Top}))$ provides a resolution of $X_\bullet$ and computes the homotopy groups and cohomology of locally constant étale sheaves. Thus, inspired by the results from topology mentioned in *Motivation from Topology*, we can define the étale topological type $(X_\bullet)_{\text{ét}}$ of $X_\bullet$ as essentially the inverse system of simplicial sets given by the connected component functor on hypercoverings.

For some control, we want the connected components functor

$$\pi_0 : (X_\bullet)_{\text{ét}} \longrightarrow \mathbf{Set}$$

to be well-behaved, and this is achieved if π_0 is the left adjoint to the constant scheme $\underline{(-)} : \mathbf{Set} \to (X_\bullet)_*$. This is achieved in the case that $X_\bullet$ consists of locally noetherian schemes since then every étale scheme over a locally noetherian scheme splits as a coproduct of its connected components. From now on, we will assume $X_\bullet$ is a locally noetherian simplicial scheme.

Proposition 24 ([Fri82, Proposition 4.1])**.** *Let X be a scheme with $U \to X$ an étale map from a connected scheme and $V \to X$ an étale separated map. Then any map $U \to V$ over X is completely determined by its restriction to a geometric point $\underline{x} : \mathrm{Spec}(\Omega) \to U$.*

With this result, we can index connected components of a scheme X by its geometric points.

Definition 25 ([Fri82, Definition 4.2])**.** A rigid covering $U \to X$ of a locally noetherian scheme X is a collection of pointed étale, separated maps

$$\alpha_{\underline{x}} : (U_{\underline{x}}, u_{\underline{x}}) \longrightarrow (X, \underline{x})$$

indexed by geometric points $\underline{x} \in X$ where each $U_{\underline{x}}$ is connected. A map of rigid coverings over a map $f : X \to Y$ of schemes

$$\phi : (\alpha : U \to X) \longrightarrow (\beta : V \to Y)$$

is a map $\phi : U' \to V$ over f such that $\phi \circ u_{\underline{x}} = v_{\underline{f(\underline{x})}}$.

Definition 26 ([Fri82, Definition 4.2]). If $U \to X$ and $V \to Y$ are rigid coverings over a base scheme S, we define the rigid product

$$U \times_S^R V \longrightarrow X \times_S Y$$

indexed by the geometric points $(\underline{x}, \underline{y})$ of $X \times_S Y$ of maps

$$\alpha_x \times_{\beta_y} : ((U_{\underline{x}} \times_S V_{\underline{y}})^0, u_{\underline{x}} \times v_{\underline{y}}) \longrightarrow (X \times_S Y, \underline{x} \times \underline{y})$$

where $(U_x \times_S V_y)^0$ is the connected component of $U_x \times_S V_y$ containing $u_x \times v_y$.

We use the notion of rigid coverings and rigid products for schemes to build the hypercoverings needed to compute the étale homotopy type of a simplicial scheme.

Proposition 24 implies that there is at most one map between rigid coverings. We set $\mathrm{RC}(X_\bullet)$ as the category of rigid coverings of a simplicial scheme $X_\bullet$. By definition, an object of $\mathrm{RC}(X_\bullet)$ is a map $U_\bullet \to X_\bullet$ of simplicial schemes such that $U_n \to X_n$ are each rigid coverings for each $n \geq 0$. One can show that $\mathrm{RC}(X_\bullet)$ is a cofiltered category. We now define the category of rigid hypercoverings.

Proposition 27 ([Fri82, Proposition 4.3])**.** *Let $X_\bullet$ be a locally noetherian simplicial scheme. A rigid hypercovering $U_{\bullet,\bullet} \to X_\bullet$ is a hypercovering with the property that*

$$U_{p,q} \longrightarrow (\mathrm{cosk}_{q-1}^{X_p} \mathrm{sk}_{q-1} U_{p,\bullet})_q$$

is a rigid covering for each $p, q \geq 0$. The category of rigid hypercoverings of $X_\bullet$ denoted $\mathrm{HRR}(X_\bullet)$ is cofiltered.

The Čech topological type of $X_\bullet$ is defined to be the pro-simplicial set $\Delta(X_\bullet) = \pi_0(\Delta N_{X_\bullet}) : \mathrm{RC}(X_\bullet) \to$ **sSet** by sending a rigid covering $U_{\bullet,\bullet} \to X_\bullet$ to the simplicial set $\pi_0(\Delta(N_{X_\bullet}(U_\bullet)))$ whose n-simplices are the set of connected components of the $(n+1)$-fold fiber product of U_n over X_n. This suggests the following definition of étale topological type.

Definition 28 ([Fri82, Definition 4.4]). Let $X_\bullet$ be a locally noetherian simplicial scheme. The étale topological type of $X_\bullet$ is defined to be the pro-simplicial set

$$(X_\bullet)_{\text{ét}} = \pi_0 \circ \Delta : \mathrm{HRR}(X_\bullet) \longrightarrow \mathbf{sSet}$$

sending a hypercovering $U_{\bullet,\bullet} \to X_\bullet$ to the simplicial set $\pi_0(\Delta(U_{\bullet,\bullet}))$ of connected components of the diagonal of $U_{\bullet,\bullet}$.

By definition, $(\pi_0(\Delta U_{\bullet,\bullet}))_n$ is the set of connected components of $U_{n,n}$. To make things a bit more precise, we detail the definitions in the case where a simplicial scheme $X_\bullet$ is the constant scheme associated to a scheme $Z \in \mathbf{Sch}$. This is exactly the notion of the étale homotopy type of a scheme as in [AM69].

Definition 29. The étale homotopy type of a scheme Z is defined to be the functor

$$Z_{\text{ht}} = \pi_0 \circ \Delta : \mathrm{Ho}(\mathrm{Cov}_{\text{hyp}}(\underline{Z}_\bullet)) \longrightarrow \mathrm{Ho}(\mathrm{Top}).$$

Thus, since $\mathrm{Ho}(\mathrm{Cov}_{\text{hyp}}(\underline{Z}_\bullet))$ is cofiltered[20], we conclude that Z_{ht} is a pro-object over $\mathrm{Ho}(\mathrm{Top})$, i.e., the Kan complexes up to weak homotopy!

[20] See Proposition 21.

Example 30 (Étale homotopy type of a constant simplicial scheme)**.** Let Z be a locally noetherian scheme and consider $\underline{Z}_\bullet$ to be the constant simplicial scheme associated to Z. If we view $(\underline{Z}_\bullet)_{\text{ét}}$ in $\mathrm{Pro}(\mathrm{Ho}(\mathrm{Top}))$ by applying the natural functor $\mathbf{sSet} \to \mathrm{Ho}(\mathrm{Top})$, then Z_{ht} is isomorphic to $(\underline{Z}_\bullet)_{\text{ét}}$ in $\mathrm{Pro}(\mathrm{Ho}(\mathrm{Top}))$.

Homotopy and cohomology groups For pro-simplicial sets, one can define formally the homotopy groups and cohomology groups valued in an abelian group as a $\mathrm{Pro}(\mathrm{Grp})$. Explicitly, we would have

$$H^*(\text{“}\varprojlim X_{i,\bullet}\text{”}, A) := \operatorname*{colim}_i H^*(X_{i,\bullet}, A)$$

for an abelian group A and a pro-simplicial set "$\varprojlim$"$X_{i,\bullet}$. Since the functor on $\mathbf{Ab}$-valued functors from cofiltered categories is exact, the above cohomology groups are well-behaved. The cohomology functors H^* and homotopy groups π_* on $\mathrm{Pro}(\mathbf{sSet})$ factor through $\mathrm{Pro}(\mathrm{Ho}(\mathrm{Top}))$. In other words, we have defined cohomology and the homotopy groups of the étale topological type of any simplicial scheme!

Theorem 31. *Let X be a locally noetherian scheme and A an abelian group. Then we have a canonical isomorphism*

$$H^n_{\text{ét}}(X, A) \xrightarrow{\cong} H^n(X_{\text{ht}}, A).$$

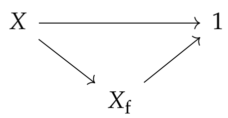
We have used implicitly that the étale homotopy type of a scheme and the étale topological type of the associated constant simplicial scheme coincide (see Example 30) in $\mathrm{Pro}(\mathrm{Ho}(\mathrm{Top}))$. Actually, one can compute cohomology of locally constant étale sheaves on X via the étale homotopy type of X (see [Fri82, Proposition 5.9]).

The homotopy groups are new invariants of a (simplicial) scheme.

Note: Model Categories in Brief

The formalism of model categories [21] emerged from the following problem. Given a category $\mathcal{C}$ and a set of morphisms ("weak equivalences") $WE \subset \mathrm{Mor}(\mathcal{C})$, one can construct the *homotopy* category $\mathrm{Ho}(\mathcal{C})$ associated to $\mathcal{C}$ by formally inverting the maps in WE. Given two objects X and Y in $\mathcal{C}$, in practice one is not able to compute the hom-set $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X, Y)$ in the homotopy category. Model categories provide a nice solution to this problem.

Very roughly, a model structure on $\mathcal{C}$ is the data of three classes of morphisms: weak equivalences, fibrations and cofibrations; each morphism satisfying lifting properties with respect to morphisms in the other classes. An object X is called *fibrant* if the terminal morphism $X \to 1$ is a fibration, and dually X is *cofibrant* if the initial morphism $0 \to X$ is a cofibration. Morphisms in model categories come with *functorial factorizations*, in particular the terminal morphism $X \to 1$ can be factored as



where the first map is a weak equivalence and the second is a fibration. The object X_{f} is therefore fibrant and called a fibrant replacement of X. The dual concept is called cofibrant replacement. The main theorem is the following.

Theorem 32 (Quillen; see [Hov99, Theorem 1.2.10]). *Given two objects X and Y in $\mathcal{C}$, the natural morphism*

$$\mathrm{Hom}_{\mathcal{C}}(X_{cf}, Y_f)/\sim \longrightarrow \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X, Y)$$

is an isomorphism, where $\sim$ denotes the strong homotopy relationship.

Furthermore, there is a model category on pro-simplicial sets where weak equivalences are maps $f : X \to Y$ that induce isomorphisms of pro-local systems $\Pi_n X \xrightarrow{\sim} f^*\Pi_n Y$ for all $n \geq 0$, see [Isa01, Theorem 6.4]. All objects are cofibrant. This allows us to construct the homotopy category $\mathrm{Ho}(\mathrm{Pro}(\mathbf{sSet}))$, and define the étale homotopy type functor as the composite $\mathbf{Sch} \to \mathrm{Pro}(\mathbf{sSet}) \to \mathrm{Ho}(\mathrm{Pro}(\mathbf{sSet}))$.

[21] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999

The étale homotopy groups of an étale topological type

The aim of this section is to introduce the definition of the étale homotopy groups of $X_{\text{ét}}$ and their relation to Grothendieck's étale fundamental group.

Étale homotopy groups.

Let $(\mathbf{sSet})_*$ denote the category of pointed simplicial sets, and let $\mathrm{Pro}((\mathbf{sSet})_*)$ be its pro-category. A *pointed pro-space* is a pro-object $Y = \{Y_i\}_{i\in I}$ together with a compatible system of basepoints $y = \{y_i \in Y_i\}_{i\in I}$.

In our situation, the rigid construction provides natural basepoints. Concretely, for a locally noetherian simplicial scheme $X_\bullet$ and a geometric point $\underline{x}$ of $X_\bullet$, the étale topological type

$$(X_\bullet)_{\text{ét}} := (\pi^0 : \mathrm{HRR}(X_\bullet) \to \mathbf{sSet}); \quad U_{\bullet,\bullet} \mapsto \pi^0(U_{\bullet,\bullet}).$$

admits a canonical pointing above $\underline{x}$ (coming from the compatible liftings built into rigid coverings). Hence we can define the n-th homotopy group of $X_{\text{ét}}$ as follows:

Definition 33 ([Fri82, Definition 5.1])**.** Let $(X_\bullet, x)$ be a locally noetherian pointed simplicial scheme. For $n > 0$, define its n-th homotopy group to be the pro-object

$$\pi_n((X_\bullet, x)_{\text{ét}}) : \mathrm{HRR}(X_\bullet, x) \to \mathbf{Set}_*; \quad (U_{\bullet,\bullet}, u) \mapsto (\pi_n(\pi^0(U_{\bullet,\bullet}), u))$$

Thus $\pi_0((X_\bullet, x)_{\text{ét}}) \in \mathrm{Pro}(\mathbf{Set}_*)$, and $\pi_n((X_\bullet, x)_{\text{ét}}) \in \mathrm{Pro}(\mathrm{Grp}_*)$ for $n \geq 1$; moreover $\pi_n((X_\bullet, x)_{\text{ét}}) \in \mathrm{Pro}(\mathbf{Ab}_*)$ for $n \geq 2$.

For example, in the case of $U_{\bullet,\bullet} = N_X(U)$[22], we can calculate $\pi_0(\pi^0(U_\bullet))$ immediately by

$$\pi_0\big(\pi^0(N_X(U))\big) = \mathrm{coeq}\big(\pi_0(U \times_X U) \rightrightarrows^{d_0}_{d_1} \pi_0(U)\big).$$

and from the fact that the following sequence in **Set** is exact:

$$\pi_0(U \times_X U) \rightrightarrows^{d_0}_{d_1} \pi_0(U) \to \pi_0(X) \to *.$$

Corollary 34. *In the above situation, we have*

$$\pi_0\big(\pi^0(N_X(U))\big) \xrightarrow{\sim} \pi_0(X).$$

The following proposition generalizes the above Čech nerve case to any simplicial scheme:

Proposition 35 ([Fri82, Proposition 5.2])**.** *Let $(X_\bullet, x)$ be a locally noetherian pointed simplicial scheme. Then the natural morphism*

$$\pi_0((X_\bullet, x)_{\text{ét}}) \to \pi_0(X_\bullet, x).$$

is an isomorphism in $\mathrm{Pro}(\mathbf{Set}_*)$.

Relation with Grothendieck's étale fundamental groups.

The first nontrivial homotopy invariant is π_1. Let X be a connected locally noetherian scheme and x a geometric point of X. Then $\pi_1((X, x)_{\text{ét}})$ canonically identifies with Grothendieck's étale fundamental group after taking the profinite completion. Let us see this from the viewpoint of the principal G-fibrations over simplicial schemes.

Definition 36 ([Fri82, Definition 5.3])**.** Let $U_\bullet$ be a simplicial scheme and let G be a (discrete) group. A principal G-fibration over $U_\bullet$ is a map of simplicial schemes $f : U'_\bullet \to U_\bullet$ together with a right action of G on $U'_\bullet$ over $U_\bullet$ such that

1. There exists a étale surjective morphism $V \to U_0$ such that $V \times_{U_0} U'_0 \cong V \times G = \coprod_{g \in G} V$ as G-schemes over V.
2. For each simplicial structure map $\alpha : [n] \to [m]$ in Δ, the square

$$\begin{array}{ccc} U'_m & \xrightarrow{\alpha'} & U'_n \\ \downarrow f & & \downarrow f \\ U_m & \xrightarrow{\alpha} & U_n \end{array}$$

is cartesian[23].

[22] I.e., the Čech nerve of an étale surjective morphism $U \to X$ of locally noetherian schemes

[23] When $U_\bullet$ is a scheme X, a principal G-fibration (or G-torsor) over X is a map of schemes $f : Y \to X$ together with a right action of G on Y over X satisfying the first condition.

A map of G-fibrations over $U_\bullet$ is an isomorphism of simplicial schemes over $U_\bullet$ commuting with the action of G.

Let us denote by $\pi_1^{\mathrm{SGA1}}(X,x)$ and by $\pi_1^{\mathrm{SGA3}}(X,x)$ the two étale fundamental groups[24] of Grothendieck. If G is a *finite* group, then there is a natural bijection

$$\mathrm{Hom}_{\mathrm{cont}}(\pi_1^{\mathrm{SGA1}}(X,x),G) \;\cong\; \{\text{(pointed) finite étale } G\text{-torsors over } X\}/\text{isom.}$$

[24] The SGA3 fundamental group is an enlarged version of the SGA1 one, which applies to potentially singular varieties and is not profinite in general, see SGA3 Exposé X § 6.

The following Proposition is a generalization of this result to our homotopy group $\pi_1((X_\bullet,x)_{\text{ét}})$.

Proposition 37 ([Fri82, Proposition 5.6]). *Let $(X_\bullet,x)$ be a pointed, connected simplicial scheme and let G be a group. Then*[25]

$$\mathrm{Hom}(\pi_1((X_\bullet,x)_{\text{ét}}),G) \cong \{\textit{(pointed) pri. } G\textit{-fib.}/X_\bullet\}/\textit{isom.}$$

[25] We use the standard convention for morphisms out of a pro-object here: if $P=\{P_i\}_{i\in I}$ is a pro-group and G is a (discrete) group, then

$$\mathrm{Hom}(P,G) \;:=\; \operatorname*{colim}_{i\in I}\mathrm{Hom}(P_i,G).$$

In particular, if X is locally noetherian, then $\pi_1((X,x)_{\text{ét}}) \simeq \pi_1^{\mathrm{SGA3}}(X,x)$, see also [AM69, Corollary 10.7]. When restricting to finite quotients, one thus obtains the following.

Corollary 38. *Let X be a locally noetherian scheme with a geometric point x. Then we have*

$$\pi_1((X,x)_{\text{ét}})^\wedge \xrightarrow{\sim} \pi_1^{SGA1}(X,x).$$

Here, $(-)^\wedge$ denotes the profinite completion.

It then follows [AM69, Theorem 11.1] that for X geometrically unibranch, in particular normal, $\pi_1((X,x)_{\text{ét}}) \xrightarrow{\sim} \pi_1^{\mathrm{SGA1}}(X,x)$.

Applications to étale topological anabelian results

Anabelian Geometry

For now, let k denote a *sub-p-adic* field[26], X a smooth geometrically connected k-variety, and Y a k-hyperbolic curve. Let G_k denote the absolute Galois group of k, Π_X Grothendieck's étale fundamental group of X at a fixed basepoint x, and Δ_X the geometric fundamental group of X[27]. Then we have the following exact sequence

$$1 \longrightarrow \Delta_X \longrightarrow \Pi_X \longrightarrow G_k \longrightarrow 1.$$

[26] A subfield of a finitely generated extension of $\mathbb{Q}_p$, e.g., a number field.

[27] I.e., the étale fundamental group of $X\otimes_k \bar{k}$ at a given algebraic closure of k, with corresponding choice of geometric basepoint.

One of the central problems of anabelian geometry is whether one is able to reconstruct the (class of isomorphism of the) k-variety X given the extension above. The following result [28] was established by S. Mochizuki following numerous contributions of anabelian geometers such as H. Nakamura, A. Tamagawa[29].

[28] Shinichi Mochizuki. The local pro-p anabelian geometry of curves. *Invent. Math.*, 138(2):319–423, 1999

[29] For a general survey and references, we refer again to [NTM01]

Theorem 39. *For X a smooth geometrically connected variety and Y a hyperbolic curve over k (a sub-p-adic field), the canonical map*

$$\mathrm{Hom}_k^{dom}(X,Y) \xrightarrow{1:1} \mathrm{Hom}_{G_k}^{open}(\Pi_X,\Pi_Y)_{\Delta_Y}$$

is a bijection, where the Δ_Y on the right side means that we are taking orbits under the Δ_Y action.

A homotopical reformulation

The homotopical reformulation of Theorem 39 results from the construction of the following diagram and the property of the dashed arrow $(-)_{\text{ét}}$:

$$\begin{array}{ccc} \mathrm{Hom}_k^{\mathrm{dom}}(X,Y) & \xrightarrow{1:1} & \mathrm{Hom}_{G_k}^{\mathrm{open}}(\Pi_X,\Pi_Y) \\ {\scriptstyle (-)_{\text{ét}}}\downarrow & & \uparrow{\scriptstyle \pi_1(-)} \\ \mathrm{Hom}^{\pi_1-\mathrm{open}}_{\mathrm{Ho}(\mathrm{Pro}(\mathbf{sSet}))/k_{\text{ét}}}(X_{\text{ét}},Y_{\text{ét}}) & \longrightarrow & \mathrm{Hom}^{\pi_1-\mathrm{open}}_{\mathrm{Ho}(\mathrm{Pro}(\mathbf{sSet}))/(k_{\text{ét}},*)}((X_{\text{ét}},*),(Y_{\text{ét}},*))_{\Delta_Y} \end{array}$$

where π_1-open means maps that become open after applying $\pi_1(-)$, and the diagram is commutative by construction and the previous sections.

Because the absolute Galois group G_k is strongly center free, one first shows by standard homotopy theory constructions that the bottom map is a bijection, see [SS16, Proposition 2.4 - Corollary 2.6] and Appendix ibid.

We will now explain why the right-hand side map $\pi_1(-)$ is a bijection. The main idea is that one can reduce the problem to one about classifying spaces of pro-groups, which are easier to handle. This reduction is possible thanks to the following result.

Proposition 40. *For Y a curve that is affine or of genus $g > 0$, we have*[30]

$$\pi_n((Y,y)_{\text{ét}}) = 0 \qquad \textit{for } n \geq 2.$$

[30] A general space with this property is called an *étale* $K(\pi,1)$.

A proof can be found in [Extensions with restricted ramification and duality for arithmetic schemes] Schmidt, Prop. 15, and is based on étale cohomology computations.

One consequence is that we have a weak equivalence $(Y,y)_{\text{ét}} \sim B\pi_1((Y,y)_{\text{ét}})$. We now explain what the right-hand side term means.

The nerve functor $N : \mathbf{Cat} \to \mathbf{sSet}, \mathcal{C} \mapsto \mathrm{Hom}(-,\mathcal{C})$ maps a category to a simplicial set. When G is a discrete group, the nerve of the category with one object and morphism group G, denoted BG, has the following nice description. The 0-simplex is the canonical basepoint, the 1-simplices are the elements of G, and 2-simplices are formed by "completing triangles", i.e. if $f,g \in G$, then gf is a 2-simplex as in Figure 1.

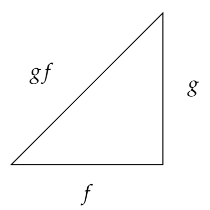


Figure 1: A 2-simplex in BG

In general, n-simplices are just n-tuples of elements of G. The association $G \to BG$ is a functor, and we can talk about BG for pro-groups G. That's what we'll do now.

Proposition 41. *Let G denote a pro-group. The space BG enjoys the following property:*

$$\pi_n(BG,*) = \begin{cases} G & n = 1 \\ 0 & n \geq 2 \end{cases}$$

The canonical morphism $G = BG_1 \to \pi_1(BG) = G$ that associates to a 1-simplex its corresponding class in the fundamental group is the identity. For two pro-groups G, G', this allows us to define the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Pro(Grp)}}(G, G') & \xrightarrow{B(-)} & \mathrm{Hom}_{\mathrm{Pro}(\mathbf{sSet})}(BG, BG') \\ \| & \swarrow & \downarrow \\ \mathrm{Hom}_{\mathrm{Pro(Grp)}}(G, G') & \xleftarrow{\pi_1(-)} & \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Pro}(\mathbf{sSet}))}(BG, BG') \end{array}$$

Because the image of an n-simplex is determined by the images of its edges in dimension one, and $BG_1 = G$, the anti-diagonal map is injective. To prove that the bottom map is bijective, it is therefore sufficient to prove that the right-hand side map is surjective[31]. While not true in general, we can actually reduce the general case to the case where BG' is fibrant by using a well-chosen fibrant replacement of BG'.

[31] It would be true if BG' was a fibrant object, by the first theorem on model categories.

Establishing that the map

$$\mathrm{Hom}^{\pi_1-\mathrm{open}}_{\mathrm{Ho}(\mathrm{Pro}(\mathbf{sSet}))/k_{\text{ét}}}(X_{\text{ét}}, Y_{\text{ét}}) \to \mathrm{Hom}^{\mathrm{open}}_{G_k}(\Pi_X, \Pi_Y)$$

is bijective then follows from a reduction to the bijection shown above.

We can finally deduce the homotopical reformulation of the classical Grothendieck conjecture, that is:

Theorem 42 ([SS16, Theorem 3.2])**.** *With the notations and under the assumptions of this section, the canonical map*

$$\mathrm{Hom}^{dom}_k(X, Y) \xrightarrow{1:1} \mathrm{Hom}^{\pi_1-open}_{\mathrm{Ho}(\mathrm{Pro}(\mathbf{sSet}))/k_{\text{ét}}}(X_{\text{ét}}, Y_{\text{ét}})$$

is a bijection.

Two anabelian applications

Let now k be a number field, and X, Y smooth and geometrically connected k-varieties that can be embedded as a locally closed subscheme of a product of k-hyperbolic curves. One can further establish [32]:

[32] The proof of the following theorem was not discussed during the talk; we refer to [SS16].

Theorem 43 ([SS16, Theorem 4.7])**.** *The canonical morphism*

$$\mathrm{Isom}_k(X, Y) \longrightarrow \mathrm{Isom}_{k_{\text{ét}}}(X_{\text{ét}}, Y_{\text{ét}})$$

admits a functorial retraction.

For a special kind of good fibration into hyperbolic curves (*à la SGA4*), the above map is a bijection, see [SS16, Theorem 7.2].

By a result of Artin, such fibrations appear as neighborhoods of every point in smooth varieties over number fields[33]. This yields the following result.

[33] The existence of such good neighborhoods over an algebraically closed field is given in SGA4, XI 3.3.

Theorem 44. *Every point of a smooth and geometrically connected variety over a number field has a Zariski-open fundamental system of neighborhoods consisting of anabelian varieties.*

Here anabelian means that there is a canonical bijection between isomorphisms of two elements of the neighborhood and their étale homotopy types. We refer to [SS16, § 6 and Theorem 1.5] for the homotopical approach[34].

[34] A more general result over sub-p-adic fields is given by [Hos20, Theorem A] with classical techniques, also with an absolute version, see Theorem C ibid.

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