Notes from "Atelier de Géométrie Arithmétique" 数論幾何学のアトリエ[1]

Around the Grothendieck-Teichmüller group

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[2] Based on the talks of L. Dauger, N. Takada, A. Assoun and N. Yamaguchi.

The Grothendieck-Teichmüller group, whose principle was originally introduced by Grothendieck, has been a center of attention since its formal introduction[3] by Drinfel'd and Ihara. Classical work of Ihara, Lochak, Nakamura and Schneps have since developed the original program towards $\widehat{GT}$ as a possible combinatorial description of $G_{\mathbf{Q}}$. The recent progress[4] of Hoshi, Minamide and Mochizuki, on the group theoretic reconstruction of various invariants of hyperbolic curves, shed a new light on the group $\widehat{GT}$ as an anabelian object.

[3] V. G. Drinfel'd. On quasitriangular quasi-Hopf algebras and on a group that is closely connected with $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. *Algebra i Analiz*, 2(4):149–181, 1990

[4] Yuichiro Hoshi, Arata Minamide, and Shinichi Mochizuki. Group-theoreticity of numerical invariants and distinguished subgroups of configuration space groups. *Kodai Math. J.*, 45(3):295–348, 2022

The goal of this *Atelier* was to give a complete description of $\widehat{GT}$ as given by Ihara and introduce the anabelian tools in order to apprehend the proof of the recent progress of Hoshi, Minamide and Mochizuki. These notes give an overview of the work done during this one-day event, see the program[5] for more context and the abstracts of the talks.

[5] The program and all the information regarding this session of the *Atelier* can be found here : AHGT website.

An arithmetic, combinatorial description of $\widehat{GT}$

The introduction of the group $\widehat{GT}$

The theoretical objective of the Grothendieck-Teichmüller theory, as sketched[6] in "Esquisse d'un programme", is in broad terms, to provide a combinatorial description of $G_{\mathbf{Q}}$ in terms of geometric invariants. The main idea in order to achieve this is to study $G_{\mathbf{Q}}$ through its action on a certain tower $\{\mathcal{M}_{g,m}\}_{g,m}$ of moduli spaces of curves.

[6] Alexandre Grothendieck. Esquisse d'un programme. In *Geometric Galois actions, 1*, volume 242 of *London Math. Soc. Lecture Note Ser.*, pages 5–48. Cambridge Univ. Press, Cambridge, 1997b

Grothendieck-Teichmüller theory can be seen as a triangle between arithmetic, geometry and combinatorics as illustrated below

$$\begin{tikzcd} G_{\mathbf{Q}} \arrow[r, hook] \arrow[dr] & \operatorname{Out}\pi_1(\mathcal{M}_{g,m}\otimes\overline{\mathbf{Q}}) \\ & \widehat{GT} \arrow[u, dashed] \end{tikzcd}$$

where the vertical arrow has been so far defined[7] for $g = 0$.

[7] Yasutaka Ihara and Makoto Matsumoto. On Galois actions on profinite completions of braid groups. In *Recent developments in the inverse Galois problem (Seattle, WA, 1993)*, volume 186 of *Contemp. Math.*, pages 173–200. Amer. Math. Soc., Providence, RI, 1995

The definition of the Grothendieck-Teichmüller group is as follows. Let F_2 be the free group on two generators x and y and $\widehat{F}_2$ its profinite completion. We also denote by G' the derived subgroup of G. We first consider the product $\widehat{Z}^{\times} \times \widehat{F}_2{}'$ endowed with the composition law[8] given by

$$(\lambda, f) \cdot (\mu, g) = (\lambda \cdot \mu, f \cdot F_{(f,\lambda)}(g)$$

[8] This law corresponds to the Galois action in the first line of the diagram above.

where $F_{(f,\lambda)}$ is the automorphism of $\widehat{F}_2$ defined by $x \mapsto x^\lambda$ and $y \mapsto f^{-1}y^\lambda f$.

Definition 1. The Grothendieck-Teichmüller[9] group $\widehat{GT}$ is defined as the subgroup of $\widehat{Z}^\times \times \widehat{F}_2'$ of tuples (λ, f) satisfying the following relations[10]

I. $f(x,y)f(y,x) = 1$

II. $f(z,x)z^n f(y,z)y^n f(x,y)x^n = 1$ for $n = \frac{\lambda-1}{2}$ and $xyz = 1$

III. $f(x_{12}, x_{23})f(x_{34}, x_{45})f(x_{51}, x_{12})f(x_{13}, x_{34})f(x_{45}, x_{31}) = 1$

where the last equation takes place in the profinite completion of the pure braid group $\widehat{P}_5$.

Theorem 2 (Ihara, Matsumoto, Drinfel'd). *There is an injection $G_{\mathbf{Q}} \hookrightarrow \widehat{GT}$ compatible with the action of $G_{\mathbf{Q}}$ on the outer automorphism group of the étale fundamental group of $\mathbf{P}^1_{\mathbf{Q}} \setminus \{0, 1, \infty\}$.*

More precisely, there is a $\widehat{GT}$-action[11] on the genus 0 tower $\{\mathcal{M}_{0,m}\}_m$ that extends and is compatible with the Galois one on $\mathcal{M}_{0,4} \simeq \mathbf{P}^1 \setminus \{0, 1, \infty\}$.

The central conjecture of Grothendieck-Teichmüller theory is that this injective map is actually an isomorphism, as seen in the first paragraph of a survey[12] on the open problems in this theory.

From Geometric Galois Actions to $\widehat{GT}$

The main way to obtain geometric Galois actions from algebraic geometry is through the étale homotopy exact sequence. For an algebraic variety X over a field k it is the short exact sequence[13]

$$1 \longrightarrow \pi_1^{et}(X_{\overline{k}}, \overline{x}) \longrightarrow \pi_1^{et}(X, \overline{x}) \longrightarrow G_k \longrightarrow 1.$$

Remark that, by conjugation on the middle term, such an exact sequence yields a map[14]

$$\varphi_X \colon G_k \longrightarrow \operatorname{Out} \pi_1^{et}(X_{\overline{k}}, \overline{x})$$

which is the geometric Galois action we were looking for.

If k is a number field, by analytification, we obtain that $\pi_1^{et}(X_{\overline{k}})$ is the profinite completion of the topological fundamental group $\pi_1(X(\mathbf{C}))$. From this we deduce that for $X = \mathbf{P}^1_{\mathbf{Q}} \setminus \{0, 1, \infty\}$ we have $\pi_1^{et}(X_{\overline{k}}) = \widehat{F}_2$. The choice of a tangential basepoints[15] provides a lift of the outer action into an actual action of $G_{\mathbf{Q}}$ on $\widehat{F}_2$. We obtain the following.

Theorem 3. *The map $G_{\mathbf{Q}} \to \operatorname{Aut} \widehat{F}_2$ is injective. Any $\sigma \in G_{\mathbf{Q}}$ gives rise to an element of $\operatorname{Aut} \widehat{F}_2$ defines as follows[16]:*

[9] This version should sometimes be called the profinite Grothendieck-Teichmüller group. As we will see later pro-Σ variants can be defined and it as well as a pro-unipotent variant which is proven to be motivic.

[10] For a profinite group G and $a, b \in G$ the notation $f(a, b)$ is to represent the element of G defined by f by substituting x, y to a, b.

[11] Explicitly given for $F = (\lambda, f)$ on the classical braid generators σ_i by

$$F(\sigma_i) = f^{-1}(\sigma_i^2), y_i) \cdot \sigma_i^\lambda \cdot f(\sigma_i^2), y_i).$$

[12] Pierre Lochak and Leila Schneps. Open problems in Grothendieck-Teichmüller theory. In *Problems on mapping class groups and related topics*, volume 74 of *Proc. Sympos. Pure Math.*, pages 165–186. Amer. Math. Soc., Providence, RI, 2006

[13] The notation $X_{\overline{k}}$ is for the base change of X to $\overline{k}$, also note that G_k is the étale fundamental group of $\operatorname{Spec} k$.

[14] Note that, as before, we can make pro-Σ variants of this action by considering the maximal pro-Σ quotient of $\pi_1^{et}(X_{\overline{k}})$. In the case of abelian varieties and $\Sigma = \{\ell\}$ we recover the usual Galois action on the ℓ-adic Tate module.

[15] For a quick, in context, definition, a tangential basepoint on $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ is a map $\mathbf{Q}(t) \to \mathbf{Q}((T))$ where t denotes a choice of variable for the function field of $\mathbf{P}^1$.

[16] Here, χ is the cyclotomic character $G_{\mathbf{Q}} \to \widehat{Z}^\times$.

- $x \mapsto x^{\chi(\sigma)}$
- $y \mapsto f_\sigma(x,y) y^{\chi(\sigma)} f_\sigma^{-1}(x,y)$

such that $(\chi(\sigma), f_\sigma(x,y))$ *belongs to* $\widehat{GT}$. *Moreover, the map* $\mathbf{G_Q} \to \widehat{GT}$ *given by* $\sigma \mapsto (\chi(\sigma), f_\sigma)$ *is injective.*

The proof relies essentially on the use of Deligne's tangential basepoints[17] for $\mathbf{P}^1 \setminus \{0, 1, \infty\}$. In particular, one has to relate the actions coming from the different tangential points by using analytic continuation along certain paths[18,19].

Interlude on the log-geometry of curves

Log-geometry gives a functorial framework in order to work with stable curves and their moduli spaces. Let $\mathcal{M}_{g,m}$ be the moduli space of smooth proper curves with m marked points and genus g such that[20] $2g - 2 + m \geq 1$. The compactification $\overline{\mathcal{M}_{g,m}}$ of $\mathcal{M}_{g,m}$ is the moduli space of stable curves of genus g and m marked points[21]. These curves have at most nodal singularities. On the other hand, there is a notion of smooth log-curves to which is attached a corresponding moduli space $\mathrm{L}\mathcal{M}_{g,m}$. The main result of F. Kato[22] we are concerned with is the isomorphism[23]

$$\mathrm{L}\mathcal{M}_{g,m} \simeq \overline{\mathcal{M}_{g,m}}^{log}$$

where $\overline{\mathcal{M}_{g,m}}$ is given a log-structure which we will now introduce.

The notion of a log-scheme is that of a scheme X equipped with a sheaf of monoids[24] M_X which we call log-structure. It is required, by the definition, that there is a map of monoids[25] $\alpha\colon M_X \to \mathcal{O}_X$ which verifies that[26] $\alpha^{-1}(\mathcal{O}_X^\times) \simeq \mathcal{O}_X^\times$. It is thus clear that we have an inclusion of categories from the category of schemes to the category of log-schemes by considering the trivial log-structure on X given by $M_X = \mathcal{O}_X^\times$. A thorough treatment of log-structures can be found in "Logarithmic structures of Fontaine-Illusie" by K. Kato[27].

Exemple 4. There are four basic examples of log-schemes.

(i) For a scheme X, the trivial log-structure is given by the inclusion $\alpha\colon \mathcal{O}_X^\times \hookrightarrow \mathcal{O}_X$.

(ii) For X a regular scheme with normal crossings divisor $D \subset X$ and $j\colon X \setminus D \hookrightarrow X$ the corresponding open immersion $\alpha\colon \mathcal{O}_X \cap j_*\mathcal{O}_{X\setminus D}^\times$ defines a log-structure on X. The sheaf $M_X = \mathcal{O}_X \cap j_*\mathcal{O}_{X\setminus D}^\times$ can be described locally by the set $\{g \in \mathcal{O}_X \mid \forall x \in X \setminus D,\ g_x \in \mathcal{O}_{X,x}^\times\}$.

[17] These tangential basepoints are denoted $\vec{ij}$ where $i, j \in \{0, 1, \infty\}$. For example, $\vec{10}$ is defined by $T \mapsto 1 - T$

[18] For reference, one of them is Deligne's "le droit chemin" : $p\colon \vec{01} \to \vec{10}$.

[19] Yasutaka Ihara. On the embedding of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ into $\widehat{\mathrm{GT}}$. In *The Grothendieck theory of dessins d'enfants (Luminy, 1993)*, volume 200 of *London Math. Soc. Lecture Note Ser.*, pages 289–321. Cambridge Univ. Press, Cambridge, 1994. With an appendix: the action of the absolute Galois group on the moduli space of spheres with four marked points by Michel Emsalem and Pierre Lochak

[20] This so called hyperbolicity condition is equivalent for curve to have a non-abelian or anabelian (*i.e*, far from abelian) étale fundamental group

[21] These moduli spaces have the structure of Deligne-Mumford stacks over $\mathbf{Q}$.

[22] Fumiharu Kato. Log smooth deformation and moduli of log smooth curves. *Internat. J. Math.*, 11(2):215–232, 2000

[23] It is, at least, clear from this isomorphism that studying log-curves is already equivalent to studying hyperbolic curves and gives a different viewpoint on those objects.

[24] It is important that this is a sheaf on the étale site X_{et} and not just a regular sheaf on the topological space of X.

[25] Here, $\mathcal{O}_X$ is considered with the monoidal structure from multiplication.

[26] The notion of pre-log structure is obtained by forgetting this last condition

[27] Kazuya Kato. Logarithmic structures of Fontaine-Illusie. In *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*, pages 191–224. Johns Hopkins Univ. Press, Baltimore, MD, 1989

(*iii*) For a field k there is a pre-log structure given by

$$\begin{array}{ccc} \mathbf{N} & \longrightarrow & k \\ a & \longmapsto & 0^a \end{array}$$

for which the associated log-structure is given by $\mathbf{N} \oplus k^\times \to k$.

(*iv*) For the nodal curve $\operatorname{Spec} k[x,y]/(xy)$ there is a pre-log structure given by

$$\begin{array}{ccc} \mathbf{N}^2 & \longrightarrow & k[x,y]/(xy) \\ (a,b) & \longmapsto & x^a y^b \end{array}$$

The property of log-smoothness is given as follows.

Definition 5. A morphism of log-schemes $f\colon X \to Y$ is said to be smooth if it is locally of finite presentation and if for every closed immersion of affine log-schemes $T \to T'$ defined by a square zero ideal such that the following square commutes

$$\begin{tikzcd} T \arrow[r] \arrow[d] & X \arrow[d] \\ T' \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

there is, étale-locally[28], a map of log-schemes $T' \to X$ (represented by the dashed arrow) that makes the diagram commutes.

It can be shown that the nodal curve $\operatorname{Spec} k[x,y]/(xy)$ with its log-structure defined before is log-smooth using the criterion[29] given by Theorem 3.5 of *ibid*. A much more general statement is that *every nodal curve can be equipped with a structure of a log-smooth curve*.

Definition 6. A log-curve over a log-scheme S is a log-smooth and integral map of fine and saturated[30] log-schemes $f\colon X \to S$ such that the scheme theoretic geometric fibers of f are reduced and connected curves.

A structure result for log-curves over fields is as follows.

Theorem 7. *Let k be a separably closed field and X a log-curve over k. Then*

1. *The underlying scheme of X has at most ordinary double points*[31].
2. *There exists distinct points $s_1, \dots, s_n$ of the smooth locus of X such that*

$$M_X \simeq \mathbf{Z}_{r_1} \oplus \cdots \oplus \mathbf{Z}_{r_m} \oplus \mathbf{N}_{s_1} \oplus \cdots \oplus N_{s_n}$$

where $\{r_1, \dots, r_m\}$ is the set of double points of X.

On the other hand F. Kato shows, as alluded before, that there is a canonical structure of log-curve for stable curves. From this we can equip $\overline{\mathcal{M}_{g,m}}$ with a canonical structure of a log-stack and prove the desired isomorphism $\mathrm{L}\mathcal{M}_{g,m} \simeq \overline{\mathcal{M}_{g,m}}^{log}$.

[28] At first glance, the second part of this definition seems to be the identical to the definition of formal smoothness, but here the existence of the lift is only local for the étale topology, which is a much weaker condition.

[29] Kazuya Kato. Logarithmic structures of Fontaine-Illusie. In *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*, pages 191–224. Johns Hopkins Univ. Press, Baltimore, MD, 1989

[30] This can be seen as an analogue of the notion of coherent module for the log-structure.

[31] That is étale-locally given by the nodal curve $k[x,y]/(xy)$.

Some tools of anabelian geometry

For X an algebraic variety over a field k recall that we have the fundamental exact sequence

$$1 \longrightarrow \pi_1^{et}(X_{\bar{k}}) \longrightarrow \pi_1^{et}(X) \longrightarrow G_k \longrightarrow 1.$$

The basic problem of anabelian geometry is to understand how to recover X from the profinite group $\pi_1^{et}(X_{\bar{k}})$ equipped with the G_k-outer action coming from the exact sequence. Grothendieck famously stated[32] that it should be possible to recover X from this group theoretical data in the case X is "anabelian" without giving a proper definition of this term[33].

[32] Alexander Grothendieck. Brief an G. Faltings. In *Geometric Galois actions, 1*, volume 242 of *London Math. Soc. Lecture Note Ser.*, pages 49–58. Cambridge Univ. Press, Cambridge, 1997a. With an English translation on pp. 285–293

[33] However, he gave as examples of anabelian spaces the three following type of spaces : hyperbolic curves, configuration spaces of points over hyperbolic curves and moduli spaces of hyperbolic curves.

More precisely, for anabelian schemes X_1 and X_2 over k we are looking for a natural bijection[34]

$$\mathrm{Isom}_{Sch/k}(X_1, X_2) \simeq \mathrm{Isom}_{G_k}(\pi_1(X_1), \pi_1(X_2))/conj.$$

[34] This, which was known as the Grothendieck conjecture for hyperbolic curves have been resolved in the 90's by the work of H. Nakamura (genus 0 curves and reconstruction of cuspidal inertia subgroups), A. Tamagawa (affine curves) and S. Mochizuki (for proper curves).

Anabelian reconstruction of data for hyperbolic curves

Let X be a hyperbolic curve over a number field k. That is X is an open subscheme of a proper smooth curve $\overline{X}$ and the divisor $\overline{X} \setminus X$ is of degree r such that $2g - 2 + r \geq 1$ where g is the genus of $\overline{X}$. Our goal in this section is to explain how to recover (g, r) from the group theoretic data of $\pi_1(X_{\bar{k}})$ with its G_k-action.

Let us denote by $\Pi_{g,r}$ the group given by the presentation

$$\langle a_1, \ldots, a_g, b_1, \ldots, b_g, s_1, \ldots, s_r \mid \prod_i [a_i, b_i] \prod_i s_i = 1 \rangle.$$

Then $\pi_1(X_{\bar{k}})$ is the profinite completion of $\Pi_{g,r}$ and its abelianization is given by $\widehat{Z}^{2g+r-\epsilon}$ where $\epsilon = 0$ if $r = 0$ and $\epsilon = 1$ otherwise. Then by considering the action of the Frobenius at a place of good reduction of X we can recover r as the rank of the subspace where the Frobenius acts by weight 2, then g can be recovered subsequently. More precisely, for a curve X over a finite field k, one can recover r by the equality $r = \mathrm{Card}\{a \in \mathcal{A} \mid |a| = \mathrm{Card}\, k\}$ where $\mathcal{A}$ is the set of roots of the characteristic polynomial of the Frobenius acting on the abelianization of $\pi_1(X_{\bar{k}})$. The genus g is recovered as the cardinality of the set $\{a \in \mathcal{A} \mid |a| = (\mathrm{Card}\, k)^{\frac{1}{2}}\}$.

S. Mochizuki, see[35] lemma 1.3.8 and 1.3.9 of "The absolute anabelian geometry of hyperbolic curves", extended these results to the setting of p-adic fields for absolute anabelian geometry. One example is the following theorem.

[35] Shinichi Mochizuki. The absolute anabelian geometry of hyperbolic curves. In *Galois theory and modular forms*, volume 11 of *Dev. Math.*, pages 77–122. Kluwer Acad. Publ., Boston, MA, 2004

Theorem 8. *Let X_1/K_1 and X_2/K_2 be hyperbolic curves over p-adic fields with genus g_1, g_2 and r_1, r_2 marked points. If there is an isomorphism $\alpha\colon \pi_1(X_1) \to \pi_1(X_2)$ then K_1 and K_2 have isomorphic residue fields and $(g_1, r_1) = (g_2, r_2)$.*

The introduction of log-geometry

We now consider hyperbolic curves over an algebraically closed field k equipped with the log-structure that was defined previously. For such a hyperbolic curve X, let the n-th configuration space be defined as

$$X_n = X \times_k X \cdots \times_k X \setminus \Delta$$

where Δ is the weak diagonals in the product. For the rest of these notes the references will be with regards to "Group- theoreticity of numerical invariants and distinguished subgroups of configuration space groups" by Hoshi, Minamide and Mochizuki[36].

Definition 9 (Definition 1.1 of *ibid*)**.** A point $x_n \in X_n$ is said to be log-full if for every geometric point x over x_n we have that

$$\dim \mathcal{O}_{X,x}/I(x, M_X) = 0$$

where $I(x, M_X)$ is the ideal generated by the image of M_X minus $\mathcal{O}_{X,x}^\times$ in $\mathcal{O}_{X,x}$.

For our purpose the interest of log-full points comes from the fact that, for such a point $x_n \in X_n$, the kernel of the natural map of log-fundamental groups[37]

$$\pi_1^{log}(x_n) \to \pi_1^{log}(\operatorname{Spec} k)$$

is isomorphic[38] to $\widehat{Z}^n$. This can be used to recover n from the group theoretic data[39] of the étale fundamental group $\pi_1(X_n)$. Precisely for ℓ a prime number different from the characteristic of k, the integer n is given[40] by

$$n = \max\{s \in \mathbf{N} \mid \exists H \subset \pi_1^\ell(X_n) \text{ closed and } H \simeq \mathbf{Z}_\ell^s\}.$$

From this, Hoshi, Minamide and Mochizuki show that one can reconstruct the map $\pi_1(X_n) \to \pi_1(X)$ and, when $r \geq 1$, and one can also recover (g, r) by the formula[41]

$$g = \frac{1}{2} \operatorname{rk}_{\mathbf{Z}_\ell} \pi_1^\ell(X)^{ab} - \operatorname{rk}_{\mathbf{Z}_\ell} \operatorname{Ker}(\pi_1^\ell(X)^{ab} \to \pi_1^\ell(\overline{X})^{ab}).$$

A new description of $\widehat{GT}$ through combinatorial anabelian geometry

Consider[42] $X = \mathbf{P}^1 \setminus \{0, 1, \infty\}$, X_n the n-th configuration space of X

[36] Yuichiro Hoshi, Arata Minamide, and Shinichi Mochizuki. Group-theoreticity of numerical invariants and distinguished subgroups of configuration space groups. *Kodai Math. J.*, 45(3):295–348, 2022

[37] There is an analogue of the fundamental exact sequence for log-schemes and their log-fundamental groups which is defined through the category of log-étale covers in the same way as the étale fundamental group.

[38] See Proposition 1.3 of *ibid*.

[39] Note that here we work over an algebraically closed field and we do not have access to the Galois action as before.

[40] See Theorem 1.6 of *ibid*.

[41] A complete reconstruction procedure for every hyperbolic (g, r) is given in Theorem 2.5 of *ibid*.

[42] Most of the anabelian results presented by Hoshi Minamide and Mochizuki work with any hyperbolic curve but for the sake of clarity we will restrict ourselves to this case in these notes.

as before and denote by Π_n its étale fundamental group. In order to understand the remarkable isomorphism

$$\widehat{GT} \times \mathfrak{S}_5 \simeq \operatorname{Out} \Pi_2$$

we first need to recall some results regarding the moduli spaces of curves of genus 0, the configuration spaces X_n and $\widehat{GT}$. First of all, for all $n \geq 0$ we have the following isomorphism[43]

$$X_n \simeq \mathcal{M}_{0,n+3}.$$

From this isomorphism we obtain a natural faithful action of $\mathfrak{S}_{n+3}$ on X_n by its action on the set of marked points[44]. This provides us with a natural inclusion $\mathfrak{S}_{n+3} \hookrightarrow \operatorname{Out} \Pi_1(X_n)$. We also define the *fiber subgroups* of Π_n to be the kernels of the standard projections $X_n \to X_m$ for any choice of $m \leq n$. By choosing the projections that forgets the factors in canonical order we get a so called standard sequence of surjections

$$X_n \to X_{n-1} \to \cdots \to X_m \to \cdots \to X_1 \to X_0$$

from which we obtain a filtration of Π_n by fiber subgroups $(K_i)_{1 \in \{0,\ldots,n\}}$, called the *standard fibration* on Π_n

$$\{1\} = K_n \subset K_{n-1} \subset \cdots \subset K_m \subset \cdots \subset K_1 \subset K_0 = \Pi_n.$$

Note that by the hyperbolicity condition we also have that quotients of subsequent terms K_{m+1}/K_m in the filtration are isomorphic to the fundamental group of a curve of type $(0, m+3)$. There is thus a well defined notion of cuspidal inertia subgroups for those[45]. Now, we will define the subgroup $\operatorname{Out}^{FCS}(\Pi_n)$ of *FCS-outer automorphism* of Π_n to be composed of the automorphisms of the following types (simultaneously)[46].

(F) *The fiber-admissible automorphisms*, that is the automorphisms that fixes the fiber subgroups, *i.e.* $\alpha(H) = H$ for every fiber subgroup $H \subset \Pi_n$.

(C) *The cusp-admissible automorphisms*, that is the automorphisms that preserves the standard filtration and induces a bijection between the set of cuspidal inertial subgroups of subsequent quotients of the standard filtration.

(S) The automorphisms which commutes with the action of $\mathfrak{S}_{n+3}$, *i.e.* elements of the centralizer $Z_{\mathfrak{S}_{n+3}}(\operatorname{Out} \Pi_n)$.

We can moreover reformulate[47] the classical result of Harbater and Schneps[48], that we have the following isomorphisms, for $n \geq 2$:

[43] By convention X_0 is a point.

[44] This is specific to our choice of X, for X of type $(g, r) \notin \{(0,3), (1,1)\}$ we should only consider the standard action of $\mathfrak{S}_n$ on X_n.

[45] These are the subgroups generated by the loops around the marked points broadly speaking.

[46] See Definition 2.7 of *ibid*.

[47] In the original paper, the notation used is $\operatorname{Out}_n^{\sharp}$.

[48] David Harbater and Leila Schneps. Fundamental groups of moduli and the Grothendieck-Teichmüller group. *Trans. Amer. Math. Soc.*, 352(7):3117–3148, 2000

$$\widehat{GT} \simeq \mathrm{Out}^{FCS}(\Pi_n) \simeq \cdots \simeq \mathrm{Out}^{FCS}(\Pi_2) \hookrightarrow \mathrm{Out}^{FCS}(\Pi_1).$$

Now, Hoshi, Minamide and Mochizuki introduced the notion[49] of *generalized fiber subgroup* of Π_n which are preserved[50] by any group theoretic automorphism of Π_n. Moreover, by considering the subgroup $\mathrm{Out}^{gF}\,\Pi_n$ of gF-admissible automorphisms, that is the automorphisms that fixes the generalized fiber subgroups, one obtains[51] an exact sequence

$$1 \longrightarrow \mathrm{Out}^{gF}\,\Pi_n \longrightarrow \mathrm{Out}\,\Pi_n \longrightarrow \mathfrak{S}_{n+3} \longrightarrow 1.$$

This exact sequence results from the preservation of the subset of cardinality $n+3$ of generalized fiber subgroups of Π_n of length[52] equal to 1. Furthermore, one obtains that the natural injection $\mathfrak{S}_{n+3} \hookrightarrow \mathrm{Out}\,\Pi_n$ is a splitting of this sequence and that it actually gives a direct product decomposition

$$\mathrm{Out}^{gF}\,\Pi_n \times \mathfrak{S}_{n+3} \simeq \mathrm{Out}\,\Pi_n.$$

The last step, that is technically involved, to obtain our desired isomorphism, is to see that $\mathrm{Out}^{gF}\,\Pi_n$ is equal to $\mathrm{Out}^{FCS}\,\Pi_n$ and use the result of Harbater and Schneps. By specializing to $n = 2$, we get[53]

$$\widehat{GT} \times \mathfrak{S}_5 \simeq \mathrm{Out}\,\Pi_2.$$

Some additional developments[54] on the construction of certain closed subgroups $BGT \subset \widehat{GT}$, potentially isomorphic to $\widehat{GT}$, provides a combinatorial model $\overline{Q}_{BGT}$ of $\overline{Q}$ give a new insight to the classical question: can we expect $G_{\mathbf{Q}} \simeq \widehat{GT}$?

[49] See Definition 2.1 of *ibid*.

[50] This notion is specially introduced to recover this preservation result in the case $(g, r) \in \{(0,3),(1,1)\}$ as in the other cases the fiber subgroups are already preserved group theoretically. This group theoriticity result is where a heavy part of the anabelian techniques discussed in the previous section lies.

[51] See Corollary 2.6 of *ibid*.

[52] The length of a generalized fiber subgroup is related to the number of factors removed by the projection defining it.

[53] See Corollary 2.8 of *ibid*.

[54] Yuichiro Hoshi, Shinichi Mochizuki, and Shota Tsujimura. Combinatorial construction of the absolute galois group of the field of rational numbers. *RIMS preprint*, 1935:98, 2020

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