Notes from "Atelier de Géométrie Arithmétique" 数論幾何学のアトリエ[1]

Spaces and perfectoids

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The theory of perfectoid spaces, introduced by P. Scholze mainly in two articles[3] has immediately shown its potency by proving new cases of the monodromy-weight conjecture[4]. It has since remained at the forefront of research in arithmetic geometry through its connections to central topics such as abelian varieties and their moduli spaces as well as p-divisible groups and Shimura varieties.

The goal of this *Atelier* was to provide an introduction to Scholze's theory of perfectoid spaces with its application to the moduli space of abelian varieties. These notes give an overview of the work done during this one-day event, see the program[5] for more context and the abstracts of the talks.

[3] Peter Scholze. Perfectoid spaces. *Publ. Math. Inst. Hautes Études Sci.*, 116:245–313, 2012; and Peter Scholze. On torsion in the cohomology of locally symmetric varieties. *Ann. of Math. (2)*, 182(3):945–1066, 2015

[4] It is a conjecture by Deligne which relates the eigenspaces of the action of the Frobenius on the étale cohomology of a projective scheme over a complete discrete valuation ring with the unipotent action from the inertia group on this cohomology

[5] The program and all the information regarding this session of the *Atelier* can be found here : AHGT website

Perfectoid rings and fields

Integral perfectoid rings

Fix a prime number $p > 0$. Perfectoid geometry provides a bridge between characteristic p and characteristic 0 via the operation of *tilting*. We first define tilting for *integral perfectoid rings*.

Definition 1. An integral perfectoid ring is a topological ring A for which there exists a non-zero divisor $\pi \in A$ such that

1. the topology on A is π-adic and A is complete for this topology,
2. $p \in (\pi^p)$,
3. the Frobenius morphism $\varphi : A/(\pi) \to A/(\pi^p)$, $x \mapsto x^p$ is an isomorphism.

We then say that π is a pseudo-uniformiser of A.

Example 2. The p-adic completions of $\mathbf{Z}_p[p^{1/p^\infty}]$, $\mathbf{Z}_p[\zeta_{p^\infty}]$ and $\mathcal{O}_{\overline{\mathbf{Q}}_p}$ are integral perfectoid rings[6] , with respective pseudo-uniformisers $p^{1/p}$, $1 - \zeta_p$ and $p^{1/p}$.

[6] More generally, if C is an algebraic extension of $\mathbf{Q}_p$ such that every element of $\mathcal{O}_C$ has a pth root modulo p, and for which there is a $\pi \in \mathcal{O}_C$ such that $p \in \pi^p\mathcal{O}_C$, then $\widehat{\mathcal{O}_C}$ is integral perfectoid.

Definition 3. The tilt of an integral perfectoid ring A is the ring

$$A^\flat = \varprojlim_{\varphi} A/(p) = \{(a_i)_{i\geq 0} \in A/(p) \mid a_{i+1}^p = a_i\}.$$

Example 4. The tilt of $\mathbf{Z}_p[p^{1/p^\infty}]^\wedge$ is $\mathbf{F}_p[\![t^{1/p^\infty}]\!]^\wedge$.

The following proposition shows that the tilt of an integral perfectoid ring is still an integral perfectoid ring.

Proposition 5. *Let A be an integral perfectoid ring.*

1. *Let π be a pseudo-uniformiser of A. The natural projections*

$$\varprojlim_{\varphi} A/(\pi^p) \to \varprojlim_{\varphi} A/(\pi) \to \varprojlim_{\varphi} A/(p) = A^\flat$$

are topological ring isomorphisms. Moreover, the canonical projection $\varprojlim_\varphi A \to \varprojlim_\varphi A/(\pi^p) \simeq A^\flat$ is a homeomorphism of multiplicative monoids.

2. *There exists a pseudo-uniformiser π admitting a compatible sequence of p^nth roots $\pi^\flat \in \varprojlim_\varphi A$. The ring $A^\flat$ is integral perfectoid and $\pi^\flat$ is automatically a pseudo-uniformiser of $A^\flat$ such that*[7]

$$A^\flat/(\pi^\flat) \simeq A/(\pi).$$

[7] The invariance of $A/(\pi)$ under tilting is the crucial property behind the various tilting correspondences.

The property that $\varprojlim_\varphi A \to A^\flat$ is a homeomorphism lets us define an *untilting* map $A^\flat \to A$, which is multiplicative but not additive.

Definition 6. Let A be an integral perfectoid ring. We define the untilting map[8] as the composite of the homomorphisms of multiplicative monoids

$$\sharp : A^\flat \simeq \varprojlim_{\varphi} A \xrightarrow{\mathrm{proj}_0} A.$$

[8] One can prove that a topological ring A of characteristic p is integral perfectoid if and only if it is perfect and its topology is π-adic for some non-zero divisor π. In this case, the untilting map $A^\flat \to A$ is an isomorphism: tilting has no effect in characteristic p.

The tilting correspondence for integral perfectoid rings provides a *relative* equivalence between a perfectoid ring and its tilt, i.e. between perfectoid A-algebras and perfectoid $A^\flat$-algebras. Here, an A-algebra is *perfectoid* if it is integral perfectoid when equipped with the canonical (π-adic) topology[9].

[9] One may prove that any non-zero divisor satisfying the first two conditions of Definition 1 automatically satisfies the third, i.e. is a pseudo-uniformiser. In particular, $\pi \in A$ is also a pseudo-uniformiser of any perfectoid A-algebra.

Theorem 7 (Tilting correspondence)**.** *Tilting induces an equivalence of categories*

$$\{\textit{perfectoid algebras over } A\} \xrightarrow{\flat} \{\textit{perfectoid algebras over } A^\flat\},$$

with quasi-inverse denoted $C \mapsto C^\sharp$.

The tilting correspondence for integral perfectoid rings may be proven by deformation-theoretic arguments[10] or by means of Fontaine's $\mathbf{A}_{\mathrm{inf}}$-theory, which provides an explicit construction of the untilting functor via the ring of Witt vectors of $A^\flat$.

[10] More precisely, Scholze proved that the category of perfectoid A-algebras is equivalent to the category of flat, perfectoid *almost* (see definition 9) $A/(\pi)$-algebras, which is invariant under tilting since $A/(\pi) \simeq A^\flat/(\pi^\flat)$.

Perfectoid Tate rings

Perfectoid Tate rings are constructed from integral perfectoid rings in the same way that $\mathbf{Q}_p$ is constructed from $\mathbf{Z}_p$.

Definition 8. A perfectoid Tate ring is a topological ring[11] of the form $R = A[1/\pi]$ where A is integral perfectoid and π is a pseudo-uniformiser of A. We say that A is an integral perfectoid subring of definition of R. A homomorphism of perfectoid Tate rings is a continuous homomorphism of rings.

[11] With topology given by the basis of open neighborhoods $\{\pi^n A \mid n \in \mathbf{Z}\}$ of 0.

We wish to define tilting of perfectoid Tate rings as $R^\flat = A^\flat[1/\pi^\flat]$. The following lemma, based on Falting's *almost-mathematics*[12], ensures that the construction is well-defined.

[12] Ofer Gabber and Lorenzo Ramero. *Almost ring theory*, volume 1800 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2003. ISBN 3-540-40594-1. DOI: 10.1007/b10047. URL `https://doi.org/10.1007/b10047`

Definition 9. Let A be an integral perfectoid ring and let

$$A^{\circ\circ} = \{f \in A \mid f^n \to 0\} = \bigcup_{n \geq 0} \pi^{1/p^n} A$$

be the set of topologically nilpotent elements. We say that an A-module M is almost zero if $A^{\circ\circ} M = 0$. A homomorphism of A-modules is an almost-monomorphism (resp. epimorphism, isomorphism) if its kernel (resp. cokernel, kernel and cokernel) is almost zero.

Lemma 10. *1. Let A be an integral perfectoid ring and B a perfectoid A-algebra. Then, $A[1/\pi] \xrightarrow{\sim} B[1/\pi]$ if and only if $A \to B$ is an almost isomorphism. Moreover, this holds true if and only if $A^\flat \to B^\flat$ is an almost isomorphism.*

2. Let R be a perfectoid Tate ring. Then, the set of power bounded[13] *elements $R^\circ = \{f \in R \mid \{1, f, f^2, \ldots\}$ is bounded$\}$ is the maximal integral perfectoid subring of definition of R.*

[13] The general topological definition of boundedness is that a subset S of a topological ring A is bounded if, for every neighborhood U of 0, there is a neighborhood V of 0 such that $S \cdot U \subseteq V$. For perfectoid Tate rings, this amounts to being contained in $\pi^N A$ for some $N \geq 0$.

Combining this with our tilting correspondence for integral perfectoid rings, we obtain a tilting correspondence for perfectoid Tate rings.

Proposition 11. *Let R be a perfectoid Tate ring. Tilting induces an equivalence*

$$\{\text{perfectoid Tate rings over } R\} \simeq \{\text{perfectoid Tate rings over } R^\flat\}.$$

Perfectoid fields

A special case of interest is when the perfectoid Tate ring R is a field. In this case, we say that R is *perfectoid field*.

Definition 12. A perfectoid field is a non-archimedean valued field C such that $\mathcal{O}_C$ is integral perfectoid[14].

[14] A perfectoid field can be more simply understood as a perfectoid Tate ring which is a field.

Example 13. The completions of $\mathbf{Q}_p(p^{1/p^\infty})$, $\mathbf{Q}_p(\mu_{p^\infty})$ and $\overline{\mathbf{Q}}_p$ are perfectoid fields.

It can be proven that, if R is a perfectoid Tate ring, it is a field if and only if its tilt is. Consequently, we obtain a tilting correspondence for perfectoid field extensions. The remarkable fact that this also holds for *finite* extensions is the content of Scholze's almost purity theorem.

Theorem 14 (Almost purity and tilting correspondence)**.** *Let K be a perfectoid field.*

(i) Any finite extension of K is perfectoid.

(ii) Tilting induces an equivalence on the étale sites of K and $K^\flat$ which preserves the degree. In particular, the absolute Galois groups of K and $K^\flat$ are isomorphic.

This theorem is straightforward to establish in the special case that $K^\flat$ is algebraically closed. In general, instead of proving directly that tilting induces an equivalence $K_{\mathrm{et}} \simeq K^\flat_{\mathrm{et}}$, one proves that untilting gives $K^\flat_{\mathrm{et}} \simeq K_{\mathrm{et}}$. This has the advantage that (i) is known in characteristic p, since perfectoid is the same as perfect in this case. Using Falting's almost-mathematics[15], one shows that untilting preserves degrees.

[15] There is a more precise version of theorem 14 which asserts that these two categories are also equivalent to the category of *almost-étale* K° (resp. $K^{\flat\circ}$) algebras.

It remains to see that untilting is surjective. Since untilting preserves degrees, Galois theory implies that this is equivalent to $\sharp(K^\flat_{\mathrm{et}})$ being cofinite in K_{et}. To see that this is the case, consider the union of all untilts $N = \bigcup_{L/K^\flat \text{ finite}} L^\sharp$. On computing $\mathcal{O}_N/(\pi)$, one sees that N is dense in $Q^\sharp$, where Q is a completed algebraic closure of $K^\flat$. Since $Q^\sharp$ is algebraically closed, it follows from Krasner's lemma that N is also algebraically closed, i.e. that untilts are cofinite in K_{et}.

Abelian varieties, their Tate modules, and moduli space of polarized abelian varieties with level structure

Abelian varieties

An abelian variety over a field k is a smooth, connected and proper group variety[16]. Let A be an abelian variety over some field k with separable closure k^s. We have a nice description of the torsion subgroups of A. For any $n \in \mathbf{Z}_{\neq 0}$, write $A[n]$ for the kernel[17] of the multiplication by n map $[n]\colon P \mapsto nP$ on A.

[16] Such group varieties have a lot of nice properties, for instance they are commutative, as we should expect by their name.

[17] It is a finite and flat group scheme.

Proposition 15. *Let $n \in \mathbf{Z}_{>0}$ such that char $k \nmid n$. Then*

$$A[n](k^s) \simeq (\mathbf{Z}/n\mathbf{Z})^{2\dim A}.$$

These groups are naturally equipped with an action of $\mathrm{Gal}(k^s/k)$ which contains the rationality datum of the torsion points.

Note that the Tate module of an abelian variety is formed using such torsion subgroups.

Definition 16. Let ℓ be a prime different from the characteristic of k. The ℓ-adic Tate module[18] of A, denoted $T_\ell A$ is defined as the limit of the following inverse system of Galois modules

$$0 \xleftarrow{\times \ell} A[\ell](k^s) \xleftarrow{\times \ell} \dots \xleftarrow{\times \ell} A[\ell^n](k^s) \xleftarrow{\times \ell} \dots$$

[18] Usually just called Tate module when ℓ is clear in the context.

Considering proposition 15, we see that

$$T_\ell A \simeq \mathbf{Z}_\ell^{2 \dim A}$$

and we have an ℓ-adic Galois representation

$$\rho_{\ell^\infty} : \mathrm{Gal}(k^s/k) \longrightarrow \mathrm{Aut}_{\mathbf{Z}_\ell}(T_\ell A) \simeq \mathrm{GL}_{2 \dim A}(\mathbf{Z}_\ell).$$

The Tate module together with the representation ρ_{ℓ^∞} datum is equivalent to the data of every $A[\ell^n]$ making it a powerful tool to study abelian varieties. Note that it is also possible to define Tate modules when $\ell = \mathrm{char}\, k$ but it fails to hold as much information[19]. The better object suiting to this situation is called the ℓ-divisible group[20] or Barsotti-Tate group $A[\ell^\infty]$. Focusing on the special case when A is an abelian variety of good reduction over an algebraically closed and complete ℓ-adic field k, Scholze establishes some links between the ℓ-divisible group of the Néron model[21] $\mathcal{A}$ of A and the Hodge-Tate sequence

$$0 \longrightarrow \mathrm{Lie}(A)(1) \longrightarrow T_\ell A \otimes_{\mathbf{Z}_\ell} k \longrightarrow \mathrm{Lie}(A^\vee)^* \longrightarrow 0$$

[19] We use a slightly different definition when ℓ is the characteristic of k, see definition 10.2 of

Bas Edixhoven, Gerard Van der Geer, and Ben Moonen. *Abelian varieties*. 2020. URL `http://van-der-geer.nl/~gerard/AV.pdf`

[20] see section 10§2 of *ibid*.

[21] The abelian variety A has good reduction means that the special fiber of its Néron model is an abelian variety over the residue field.

where $A^\vee$ denotes the dual abelian variety[22] of A. Namely $\mathrm{Lie}(A)(1) \subset T_\ell A \otimes k$ is $\mathbf{Q}_\ell$-rational if and only if the special fiber of $\mathcal{A}$ is ordinary[23] i.e.

$$\mathcal{A}_{\overline{\mathbf{F}_\ell}}(\overline{\mathbf{F}_\ell}) \simeq (\mathbf{Z}/\ell\mathbf{Z})^{\dim A}.$$

[22] *cf. infra* for an intuition about dual abelian varieties.

[23] See lemmas 3.3.1 and 3.3.6 of

Peter Scholze. On torsion in the cohomology of locally symmetric varieties. *Ann. of Math. (2)*, 182(3):945–1066, 2015

The dual abelian variety

Abelian varieties benefit from the notion of duality. That is, for every abelian variety A over a field k, there is an abelian variety $A^\vee$ over k of same dimension which represents some functor[24] associated to A. Their k-points are easily described. They are the subgroup of $\mathrm{Pic}(A)$ given by

$$A^\vee(k) = \left\{ [\mathcal{L}] \in \mathrm{Pic}(A) \mid \tau_a^* \mathcal{L} \simeq \mathcal{L}\ \forall a \in A(\bar{k}) \right\}$$

where τ_a is the translation by a for all a geometric point of A. Note that the duality operation commutes with base change by some field extension of k.

[24] see definition 6.1 of

Davide Lombardo. Abelian varieties. *lecture notes from "Luxembourg Summer School on Galois representations", July 03-07, 2018.* URL `https://people.dm.unipi.it/lombardo/Notes.php`

The moduli space of polarized abelian varieties with level structure

For any S locally noetherian schemes, an abelian scheme over S is a group scheme $\pi : X \to S$ which is smooth proper and with connected geometric fibres. [25] The notion of duality passes to abelian schemes[26] and we again write $(\ )^\vee$ for the duality operation.

[25] Abelian schemes over S are to be thought as families of abelian varieties parametrized by S, they are a generalization of abelian varieties on locally noetherian bases.

[26] David Mumford, John Fogarty, and Frances Kirwan. *Geometric invariant theory*, volume 34. Springer Science & Business Media, 1994

We can now define what is called the moduli functor of polarized abelian varieties with level structure. Let g, d, n be positive integers with n at least 2. For all S locally noetherian scheme, write $\mathcal{A}_{g,d,n}(S)$ for the set of tuples $(\pi : X \to S, \omega : X \to X^\vee, \sigma_1, \dots, \sigma_{2g} : S \to X)$ such that

1. $\pi : X \to S$ is an abelian scheme with g-dimensional fibres

2. $\omega : X \to X^\vee$ is a polarization of degree d^2, that is $\omega_*(\mathcal{O}_X)$ is locally free of rank d^2 and for any geometric point s of S, there exists some $\mathcal{L}$ ample invertible sheaf of X_s such that on points[27], $\omega_s : X_s \to X_s^\vee$ sends a to $(\tau_a^* \mathcal{L} \otimes \mathcal{L}^{-1})$

3. $(\sigma_1, \dots, \sigma_{2g})$ forms a level n structure, that is $n.\sigma_i = e_X$ for every $1 \leq i \leq 2g$ where e_X is the identity and for any geometric point s of S, $(\sigma_{i,s})_{1 \leq i \leq 2g}$ forms a basis of $X_s[n]$.

[27] it amounts to ask that ω gives a polarization of abelian varieties on every geometric fibres.

Such definitions actually give a contravariant functor

$$\mathcal{A}_{g,d,n} \colon \{\text{locally noetherian schemes}\} \longrightarrow \text{Sets}$$

and when it is representable by some scheme $A_{g,d,n}$, we say that $A_{g,d,n}$ is a fine moduli scheme for the moduli functor $\mathcal{A}_{g,d,n}$.

The existence of a fine moduli scheme for $\mathcal{A}_{g,d,n}$ is covered by Mumford and Fogarty[28]. It is achieved when $n > 6^g d \sqrt{g!}$ as a quotient of some subscheme of the Hilbert scheme of $\mathbf{P}^{6^g d - 1}$.

[28] David Mumford, John Fogarty, and Frances Kirwan. *Geometric invariant theory*, volume 34. Springer Science & Business Media, 1994

Perfectoid spaces, tilts and untilts

This section will present one of the main results of P. Scholze[29], which states the tilting equivalence of étale sites of perfectoid spaces. We first explain the theory of *Huber pairs* (or *affinoid adic spaces*) in order to define *adic spaces* [30]. Perfectoid spaces that are the main objects of this section are defined as the adic spaces which locally comes form perfectoid Huber pairs. Finally, we define the tilting map of perfectoid spaces and state the tilting equivalence of étale sites.

[29] Peter Scholze. Perfectoid spaces. *Publ. Math. Inst. Hautes Études Sci.*, 116:245–313, 2012

[30] Adic spaces are obtained by gluing affinoid adic spaces in a standard way.

Adic spaces

The notion of adic spaces is a concept introduced by R. Huber[31], which is a one of the powerful approaches to non-archimedean analytic

[31] R. Huber. Continuous valuations. *Math. Z.*, 212(3):455–477, 1993

geometry alongside Berkovich geometry. The category of adic spaces contains the category of locally Noetherian formal schemes as full subcategory, and hence we can regard adic spaces as a generalization of locally Noetherian formal schemes. To explain the notion of adic spaces, first we introduce Huber pairs, which contain the notion of Tate rings.

Definition 17. A topological ring A is *Huber* (or *f-adic*) if there exists an open subring A_0 of A and finitely generated ideal I of A_0 such that $\{I^n\}_{n\geq 0}$ forms a basis of open neighborhood of $0 \in A$.

Example 18. For example, the I-adic completion of a Noetherian ring by some ideal I is a Huber ring by regarding itself as the ring of definition. For more complex example, the ring $\mathbb{Q}_p\langle \mathbf{T}\rangle$ defined by

$$\{\Sigma_J a_J \mathbf{T}^J | a_J \in \mathbb{Q}_p, a_I \to 0 \ (|J| \to \infty\}$$

is a Huber ring by regarding $\mathbb{Z}_p\langle \mathbf{T}\rangle$ as the ring of definition.

Let us say that a subset S of A is bounded if, for any $n \geq 0$, there exists $m \geq 0$ such that $I^m \cdot S \subset I^n$. We set the following subring of A:

$$A^\circ \stackrel{\text{def}}{=} \{x \in A \mid \{x^n\}_{n\geq 0} \text{ is bounded in } A\}$$

A pair (A, A^+) is a *Huber pair* if A is a Huber ring and A^+ is an open integral closed subring[32] of A such that $A^+ \subset A^\circ$. For example, if $R = A[\frac{1}{\pi}]$ is a (perfectoid) Tate ring, then (R, A) is a Huber pair with a topological nilpotent unit $\pi \in R$.[33]

[32] The ring A° is a also open integral closed subring of A. In particular, it is the largest subring of integral elements of A.

[33] The topological nilpotent unit π is referred to as a *pseudo-uniformiser.*

Next, we introduce affinoid adic spaces, which will be the local neighborhoods of adic spaces.

Definition 19. The *adic spectrum* $\mathrm{Spa}(A, A^+)$ associated to a Huber pair (A, A^+) is the set of equivalence classes[34] of multiplicative continuous valuations $|\cdot| : A \to \Gamma_{|\cdot|} \cup \{0\}$ such that $|A^+| \leq 1$ holds. Here, $\Gamma_{|\cdot|}$ is a multiplicative totally ordered abelian group generated by $\mathrm{Im}(|\cdot|)\backslash\{0\}$ with topology coming from the order.

[34] Two valuations $|\cdot|$ and $|\cdot|'$ are equivalent if there exists an ordered monoid isomorphism $\Gamma_{|\cdot|} \cup \{0\} \to \Gamma_{|\cdot|'} \cup \{0\}$ such that the following diagram is commutative:

$$\begin{array}{ccc} A & \longrightarrow & \Gamma_{|\cdot|} \cup \{0\} \\ & \searrow & \downarrow \cong \\ & & \Gamma_{|\cdot|'} \cup \{0\} \end{array}$$

This condition is equivalent to the condition that, for any $a, b \in A$,

$$|a| \leq |b| \iff |a|' \leq |b|'$$

hold.

The topology of an adic spectrum is defined by the topology generated by the following subsets: For g, $f_1, \cdots, f_r \in A$, such that $(f_1, \cdots, f_r)A \subset A$ is open, we define *rational subset* as:

$$U(\frac{\mathbf{f}}{g}) = U(\frac{f_1, \cdots, f_r}{g}) \stackrel{\text{def}}{=} \{|\cdot| \in \mathrm{Spa}(A, A^+) \mid |f_i| \geq |g| \neq 0,\ i = 1, \cdots, r\}$$

Next, we define a topological ringed presheaf of $X = \mathrm{Spa}(A, A^+)$. Since it is enough to consider sections of rational subsets when we define presheaves, we define the following:

$$(O_X(U(\frac{\mathbf{f}}{g})), O_X^+(U(\frac{\mathbf{f}}{g}))) \stackrel{\text{def}}{=} (A\langle \frac{\mathbf{f}}{g}\rangle, A^+\langle \frac{\mathbf{f}}{g}\rangle).$$

Here, $A\langle\frac{\mathbf{f}}{g}\rangle$ stands for the completion of $A[\frac{\mathbf{f}}{g}]$ with the topology whose local neighborhood of 0 is $\{I^n A_0[\frac{\mathbf{f}}{g}]\}_{n\geq 0}$, and $A^+\langle\frac{\mathbf{f}}{g}\rangle$ stands for the similar ring for the integral closure of $A^+[\frac{\mathbf{f}}{g}]$ in $A[\frac{1}{g}]$. We formally extend these to all opens of $\mathrm{Spa}(A, A^+)$ in the usual way and hence this definition makes the presheaves $O_{\mathrm{Spa}(A,A^+)}$, $O^+_{\mathrm{Spa}(A,A^+)}$ on $\mathrm{Spa}(A, A^+)$, which are called *structure presheaves* [35]. This spaces are objects in the following category:

[35] However, we need to remark that the structure presheaves are not always sheaves. If these are sheaves (we call this condition *sheafy*), then $\mathrm{Spa}(A, A^+)$ is referred to as an *affinoid adic space*.

Definition 20. Let v be the category such that

- The objects are tuple

$$(X, O_X, (|\cdot|_x)_{x\in X}),$$

  where, X is a topological space, O_X is a topological ringed sheaf, and $|\cdot|_x : O_{X,x} \to \Gamma_x \cup \{0\}$ is a multiplicative continuous valuation.

- The morphisms are families

$$(f, \{f_{\Gamma_x}\}_{x\in X}),$$

  where, $f : (X, O_X) \to (Y, O_Y)$ is a map of topological spaces with topological ringed sheaves and $f_{\Gamma_x} : \Gamma_{f(x)} \cup \{0\} \to \Gamma_x \cup \{0\}$ is a morphism for which the following diagram is commutative:

$$\begin{array}{ccc} O_{Y,f(x)} & \longrightarrow & O_{X,x} \\ \downarrow & & \downarrow \\ \Gamma_{f(x)} \cup \{0\} & \longrightarrow & \Gamma_x \cup \{0\} \end{array}$$

Let us record that a morphism $f : (A, A^+) \to (B, B^+)$ of Huber pairs is a morphism of topological rings satisfying $f(A^+) \subset B^+$. We obtain the following functor[36] as above:

[36] This functor induces a bijection of homomorphisms if B is complete.

$$\begin{array}{c} \text{(The category of sheafy Huber rings)} \\ \downarrow \\ \text{(The category of affinoid adic spaces)} \end{array}$$

We finally reached the definition of adic spaces.

Definition 21. An *adic space* is an object of v that is locally isomorphic to an affinoid adic space.

Perfectoid spaces

Perfectoid spaces are certain special adic spaces, which have a similar structure to a perfectoid Tate ring[37].

[37] Recall that a (perfectoid) Tate ring have a natural structure of a Huber pair.

Definition 22. A Huber pair (A, A^+) is *perfectoid* if the following conditions hold:

1. A° is bounded in A.
2. A has a topological nilpotent unit π and A is complete,
3. $p \in \pi^p A^\circ$,
4. the Frobenius $A^\circ/\pi A^\circ \to A^\circ/\pi^p A^\circ; x \mapsto x^p$ is an isomorphism,

Remark that (A, A^+) is perfectoid if and only if A^+ is bounded and integral perfectoid by definition. An important result on perfectoid Huber pairs is the tilting equivalence[38], which is that, if K is a perfectoid field, then the following natural functor is an isomorphism[39]:

$$\begin{array}{c}\{\text{Perfectoid Huber pairs}/K\} \\ \flat\big\downarrow\cong \\ \{\text{Perfectoid Huber pairs}/K^\flat\}\end{array}$$

[38] Lemma 6.2 of Peter Scholze. Perfectoid spaces. *Publ. Math. Inst. Hautes Études Sci.*, 116:245–313, 2012

[39] This result is a generalization of the same theorem for perfectoid Tate rings.

In order to generalize this notion of perfectoid Huber pairs to adic spaces, we state the following theorem.

Theorem 23. [40] *If (A, A^+) is a perfectoid Huber pair, then $O_{\mathrm{Spa}(A,A^+)}$ and $O_{\mathrm{Spa}(A,A^+)}$ are sheaves.*

[40] Peter Scholze. Perfectoid spaces. *Publ. Math. Inst. Hautes Études Sci.*, 116:245–313, 2012

This theorem states that the adic spectrum of a perfectoid Huber pair is an affinoid adic space. Hence we obtain that a *perfectoid space* is an object of ν that is locally isomorphic to an affinoid adic space $\mathrm{Spa}(A, A^+)$ for some perfectoid Huber pair (A, A^+).

Tilting equivalence

Finally, we introduce the tilting correspondence of perfectoid spaces, which induces the isomorphism between the étale site of a perfectoid spaces and the étale site of its tilt.

Definition 24. For a perfectoid space X over K, we define the tilt $X^\flat$ of X as the adic space over $K^\flat$ such that there exists a functorial isomorphism

$$\mathrm{Hom}_\nu(\mathrm{Spa}(A^\flat, A^{+\flat}), X^\flat) \cong \mathrm{Hom}_\nu(\mathrm{Spa}(A, A^+), X))$$

for all perfectoid Huber pair (A, A^+). If X is an affinoid perfectoid space $\mathrm{Spa}(A, A^+)$ over K we obtain that $\mathrm{Spa}(A, A^+)^\flat = \mathrm{Spa}(A^\flat, A^{\flat+})$.

The tilting induces a following isomorphism[41] of categories[42]:

$$\begin{array}{c}\{\text{Perfectoid spaces}/K\} \\ \flat\big\downarrow\cong \\ \{\text{Perfectoid spaces}/K^\flat\}\end{array}$$

[41] This isomorphism preserves the subcategory of affinoid perfectoid spaces.

[42] The natural morphism $\mathrm{Spa}(A, A^+) \to \mathrm{Spa}(A^\flat, A^{+\flat})$; $|\cdot| \mapsto |\cdot|^\flat$ is a homeomorphism and identitfes rational subsets, where $|\cdot|^\flat$ stand for the following morphism:

$$|\cdot|^\flat : A^\flat \to \Gamma_{|\cdot|} \cup \{0\};\ a \mapsto |a^\#|$$

Extending this natural morphism to perfectoid spaces, a perfectoid space X has a homeomorphism$X \to X^\flat$.

Next, let us state definition of étale maps of adic spaces:

Definition 25. A morphism of adic spaces is a finite étale morphism if it is locally a finite étale morphism of Huber pairs[43]. Additionally, a morphism $f : X \to Y$ of adic spaces is étale if, for any $x \in X$, there exists a open neighborhoods U of x and V of $y \stackrel{\text{def}}{=} f(x)$ such that

1. $f(U) \subset V$,
2. there exits a finite étale covering $p : W \to V$ and an open immersion $j : U \hookrightarrow W$ such that $p \circ j = f|_U$.

We can now define the étale site [44]X_{et} of an adic space X. This site is the category of étale morphisms $Y \to X$ with the topology generated by coverings $\{f_i : Y_i \to Y\}_{i \in I}$ such that $Y = \cup_{i \in I} f_i(Y_i)$.

Theorem 26. [45] *Let K be a perfectoid field and X a perfectoid space over K. Then tilting induces an equivalence of sites $X_{\text{et}} \cong X^\flat_{\text{et}}$, which is functorial in X.*

Towards a perfection of the moduli space of abelian varieties

In order to produce a perfection, that is to say a perfectoid version $\mathcal{A}_g(\infty)$ of the moduli space of abelian varieties[46,47] with n-torsion we shall proceed in two main steps. Those steps involve the *ordinary locus*[48] and the *good reduction locus*[49].

The first step is to produce the perfection of a formal neighborhood of the ordinary locus. It is done by using the theory of *canonical subgroups* and the Hasse invariant[50]. For an abelian variety A over a p-adic field a canonical subgroup is a closed subgroup $C_m \subset A[p^m]$ of the p^m-torsion of A which is, roughly speaking, a lift of the kernel of the Frobenius map,[51] which has its geometric generic fibers isomorphic to $(\mathbf{Z}/p^m\mathbf{Z})^g$. The existence of a canonical subgroup when the Hasse invariant is big enough is shown using deformation theory. Using a theorem of Illusie [52] the proof of the existence of such a subgroup is done by induction starting with $\text{Ker}\, F^m$ on the special fiber over R/p and lifting to $R/p^{2-2\varepsilon}$ with $\varepsilon < 1/2$ at the first step and repeating the argument afterwards.

This leads to the definition of a formal $p^{-m}\varepsilon$-thickening of the ordinary locus defined to be the space that classifies pairs $(A/S, u)$ such that A/S is an abelian scheme and u is related to the Hasse invariant of A. It comes with an open immersion into the standard moduli space of abelian varieties with p^m-torsion given by the forgetful functor. This family of spaces – for varying m – is canonically equipped with Frobenius maps going down, turning it into a projective system. Taking the limit gives the desired perfect thickening of the ordinary locus.

[43] A morphism $(A, A^+) \to (B, B^+)$ of Huber pair is finite étale if the following hold:

- $A \to B$ is a finite étale morphism as rings.
- B^+ is the integral closure of A^+ in B.
- The topology of B is equivalent to the canonical topology as a finite A-module. (i.e., $\{I^n x_1 + \cdots + I^n x_m\}_{n \geq 0}$ is a basis of topology of B, where $B = Ax_1 + \cdots + Ax_m$.)

[44] The notion of a site (or a Grothendieck topology) is a categorical extension of the notion of topological spaces. This originated from the fact that, in scheme theory, the Zariski topology is too coarse to have a satisfying cohomology theory.

[45] Peter Scholze. Perfectoid spaces. *Publ. Math. Inst. Hautes Études Sci.*, 116:245–313, 2012

[46] In his article P. Scholze treat a more general situation but we will restrict here to the case of the moduli space of abelian varieties.

[47] To be precise we should consider the Faltings-Chai compactification of such a space.

[48] The subspace of abelian varieties that have a p-divisible group of height g in their reduction.

[49] The subspace of integral points. It contains the ordinary locus.

[50] This invariant characterizes when an abelian variety in characteristic p is ordinary.

[51] More precisely it is asked that $C_m = \text{Ker}\, F^m \mod p^{1-\varepsilon}$ for $\varepsilon < 1/2$.

[52] Luc Illusie. Déformations de groupes de Barsotti-Tate (d'après A. Grothendieck). Number 127, pages 151–198. 1985. Seminar on arithmetic bundles: the Mordell conjecture (Paris, 1983/84)

In order to recover a perfection of the whole moduli space the second step consists in spreading the prefect thickening we constructed by using the $\mathrm{GSp}_{2g}(\mathbf{Q}_p)$-action on the whole moduli space. The main tool to make this possible is the Hodge-Tate period map which is first defined only on the topological spaces with source $|\mathcal{A}_g(\infty)|$[53] and target $|\,\mathrm{Fl}\,|$ where Fl is the flag variety classifying isotropic subspaces of K^{2g} with its standard symplectic structure. Namely, the Hodge-Tate period map is given by

$$|\pi_{HT}|\colon |\mathcal{A}_g(\infty)| \to |\,\mathrm{Fl}\,|$$

which associates to a polarized abelian variety A its Lie algebra[54] Lie A as a subspace of $\mathrm{T}_p(A) \otimes \mathbf{Q}_p$ which is identified to K^{2g}.

[53] This is obtained as the inverse limit of the topological spaces $\mathcal{A}_g[p^m]$ for m going to infinity.

[54] This Lie algebra has been introduced in section 2.

The crucial point is that the Hodge-Tate map is equivariant for the natural $\mathrm{GSp}_{2g}(\mathbf{Q}_p)$-action on both sides together with the fact that finitely many translates of any open subset of $|\,\mathrm{Fl}\,|$ containing the rational points[55] cover the whole flag space. Indeed, having these two results we see that $|\mathcal{A}_g(\infty)|$ itself is covered by finitely many translates of our perfectoid thickening of the ordinary locus. This finally gives its perfectoid structure to the whole moduli space.

[55] Note that abelian varieties in the ordinary locus have rational lie algebra from Lemma of loc. cit.

We also obtain that the Hodge-Tate map is a map of perfectoid spaces[56]. The study of this map and its fibers is an ongoing project[57].

[56] It should be noted that, in P. Scholze's words this map is "highly transcendental", that is outside of the realm of standard algebraic geometry.

[57] Ana Caraiani and Peter Scholze. On the generic part of the cohomology of non-compact unitary Shimura varieties. *Ann. of Math. (2)*, 199(2):483–590, 2024

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