Notes from “Atelier de Géométrie Arithmétique” 数論幾何学のアトリエ[1]

Recent progress on local-global principles and patching method in arithmetic geometry

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[2] Based on the talks of S. Philip, L. Loiseau, N. Yamaguchi and K. Goto.

By patching and gluing G-covers over a complete discretely valued field Harbater resolved the Regular Inverse Galois Problem for such fields[3] and proved the Abhyankar's conjecture[4]. A framework for patching over fields is introduced in [HH10][5], which is further developed in [HHK09][6] to produce local-global principles and various arithmetic results for function fields of curves over complete discretely valued fields. This method is then improved in [Me19][7] using Berkovic analytification.

The goal of this *Atelier* was to give a practical understanding of the patching method as given by these authors and how it leads to some arithmetic results. These notes give an overview of the work done during this one-day event, see the program[8] for more context and the abstracts of the talks.

To reflect the activity of the Atelier we include the original statement of the cited theorems.

[3] David Harbater. Galois coverings of the arithmetic line. In *Number theory (New York, 1984–1985)*, volume 1240 of *Lecture Notes in Math.*, pages 165–195. Springer, Berlin, 1987

[4] David Harbater. Abhyankar's conjecture on Galois groups over curves. *Invent. Math.*, 117(1):1–25, 1994

[5] David Harbater and Julia Hartmann. Patching over fields. *Israel J. Math.*, 176: 61–107, 2010

[6] David Harbater, Julia Hartmann, and Daniel Krashen. Applications of patching to quadratic forms and central simple algebras. *Invent. Math.*, 178(2):231–263, 2009

[7] Vlerë Mehmeti. Patching over Berkovich curves and quadratic forms. *Compos. Math.*, 155(12):2399–2438, 2019

[8] The program and all the information regarding this session of the *Atelier* can be found here : AHGT website

Patching over fields

The framework - Problem and solution

The formal framework for patching over fields, in the simple case of four fields[9], is as follows. Let F_1, F_2 and F_0 be fields that form an inverse system $\mathcal{F}$, i.e. there are injective maps $F_1 \hookrightarrow F_0$ and $F_2 \hookrightarrow F_0$. The limit of this system is a field F given by the intersection of the images of F_1 and F_2 in F_0.

[9] In general one can consider arbitrary large finite inverse system of fields.

Definition 1. A patching problem for $\mathcal{F}$ is then an inverse system of finite dimensional vector spaces V_1, V_2 and V_0 over the respective fields with F_i-linear maps $V_i \to V_0$ which become isomorphisms as F_0-linear maps after base change on the source.

Patching problems form a category with the adequate choice of morphisms[10]. It should be noted that we can consider patching problems with added structure, for instance algebras or quadratic forms. For a finite dimensional vector space V over F the base change functors give rise to a canonical patching problem, i.e. by denoting $\operatorname{Vec} K$ the category of finite dimensional K-vector spaces there is a functor $\beta\colon \operatorname{Vec} F \to \operatorname{Vec} F_1 \times_{\operatorname{Vec} F_0} \operatorname{Vec} F_2$.

[10] With our four field example this category is equivalent to the 2-fiber product of the categories of finite dimensional spaces over the respective fields.

A patching problem P is said to have a solution when it is in the image of the functor β, the solution being the finite dimensional vector space V over F such that $\beta(V) \simeq P$. The fundamental result about patching over fields, that provides existence of solutions is the following[11].

[11] David Harbater and Julia Hartmann. Patching over fields. *Israel J. Math.*, 176: 61–107, 2010

Theorem 2 ([HH10], Prop. 2.1). *Let F_1, F_2 and F_0 be fields such that $F_1 \subset F_0$ and $F_2 \subset F_0$. Let $F = F_1 \cap F_2$. Let*

$$\beta\colon \operatorname{Vec} F \longrightarrow \operatorname{Vec} F_1 \times_{\operatorname{Vec} F_0} \operatorname{Vec} F_2$$

be the natural map given by base change. Then the following statements are equivalent:

(1) β is an equivalence of categories.

(2) For every positive integer n and every matrix $A \in \mathrm{GL}_n(F_0)$ there exist matrices $A_i \in \mathrm{GL}_n(F_i)$ such that $A = A_1 A_2$.

Moreover, if these conditions hold, then the inverse of β (up to isomorphism) is given on objects by taking the fibre product.

The main idea from this theorem is that solving patching problems is the same as a matrix decomposition property[12].

[12] If we consider patching problems with added structure the equivalent matrix decomposition result should be given for a suitable linear group, such as $\mathrm{O}(q)$ for a quadratic form.

Patching over function fields of curves

The fields for which the framework of the previous section is applied appear from a geometrical context. To be precise we consider a smooth[13] projective curve $\mathcal{C}$ over a complete discrete valuation ring T with uniformizer t and residue field k. By the smoothness hypothesis the special fiber X_k of the curve is irreducible. Let η be its generic point.

[13] The smoothness hypothesis can, and should be weakened to normal, irreducible for the applications, nevertheless we will assume it here.

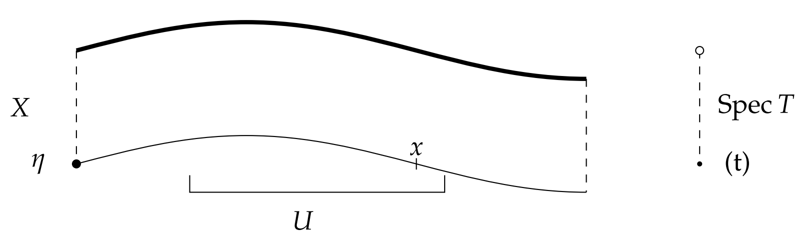


Figure 1: Curve X with generic and special fiber over Spec T

For a subset U of X_k we define a field F_U in the following way. Consider the ring R_U of regular functions on U, that is a subring of $\mathcal{O}_{X,\eta}$ and its t-adic completion $\widehat{R_U}$. This ring is a domain and we let F_U be its fraction field. The basic[14] patching result in this setting is the following[15].

[14] There are numerous variants of this result when considering different subsets of X_k and different constructions of fields with regards to these subsets.

[15] David Harbater and Julia Hartmann. Patching over fields. *Israel J. Math.*, 176: 61–107, 2010

Theorem 3 ([HH10], Theo. 4.12). *Let T be a complete discrete valuation ring and let $\widehat{X}$ be a smooth connected projective T-curve with closed fiber X.*

Let U_1, U_2 be subsets of X. Then the base change functor

$$\mathrm{Vec}(F_{U_1 \cup U_2}) \longrightarrow \mathrm{Vec}(F_{U_1}) \times_{\mathrm{Vec}(F_{U_1 \cap U_2})} \mathrm{Vec}(F_{U_2})$$

is an equivalence of categories.

From patching to local-global principles through linear algebraic groups

The local-global principles obtained from the patching theorems such as Theorem 3 concern varieties over function field of curves. We recover rational points from the existence of points over the different fields F_U[16], as given by the following local-global principle for certain homogeneous spaces[17,18].

[16] In the classical setting the base field is $\mathbf{Q}$ and the local fields involved are the completions $(\mathbf{Q}_p)_{p \text{ prime}}$ and $\mathbf{R}$.

[17] The apparition of linear algebraic groups in the picture should not be surprising in the light of the matrix decomposition property of Theorem 2.

[18] David Harbater, Julia Hartmann, and Daniel Krashen. Applications of patching to quadratic forms and central simple algebras. *Invent. Math.*, 178(2):231–263, 2009

Theorem 4 ([HHK09], Theo 3.7)**.** *Let T be a complete discretely valued ring with residue field k and X an irreducible projective T-curve with function field F. Assume given a finite map $f\colon X \to \mathbf{P}^1_T$ such that $\mathcal{P} = f^{-1}(\infty)$ contains all points at which distinct irreducible components of the closed fibre of X meet, and denote by $\mathcal{U}$ be the collection of irreducible components U of $f^{-1}(\mathbf{A}^1_k)$.*

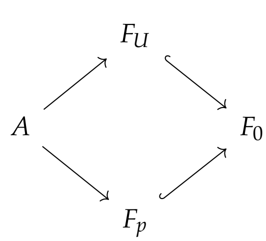
For G be a rational connected linear algebraic group over F that acts transitively on the points of an F-variety H, we have that $H(F) \neq \varnothing$ if and only if $H(F_p) = \varnothing$ for all $p \in \mathcal{P}$ and $H(F_U) \neq \varnothing$ for every $U \in \mathcal{U}$.

From the patching point of view we should look at the variety H over F as a moduli space for some algebraic structure and the linear algebraic group acting on it as the automorphism group of such structure.

A sketch of the proof, in the case of 4 fields, can be given as follows. We start with points x_U and x_p with value in the fields F_U and F_p and an overfield F_0. The transitivity of the action of the F_0-points make it such that $gx_U = x_p$ for some element $g \in G(F_0)$ [19]. The matrix decomposition result[20] gives that $g = g_p \cdot g_U$ so that by taking the elements $g_U x_U$ and $g_p^{-1} x_p$ and an affine open Spec A containing them both we get a commutative diagram

[19] This should be seen as the isomorphism condition to set up our patching problem.

[20] This step should be seen as having a solution to our patching problem.



Thus a map from A to the limit F of the inverse system given by F_U, F_p and F_0. This map gives the F-point of H that we sought.

Some applications to quadratic forms

Let us consider a finite dimensional vector space E equipped with a quadratic form q over the function field F of a curve as before. In order to apply Theorem 4 to this situation[21] we should first note that the linear algebraic group we consider is $O(q)$. The transitivity condition is then given by an application of Witt's extension theorem. We thus get the following result[22].

[21] The variety H here is the one defined by the homogeneous polynomial defining q.

[22] David Harbater, Julia Hartmann, and Daniel Krashen. Applications of patching to quadratic forms and central simple algebras. *Invent. Math.*, 178(2):231–263, 2009

Theorem 5 ([HHK09], Theo 4.2)**.** *In the context of Theorem 4, suppose q is a quadratic form of dimension unequal to 2, such that q_{F_ξ} is isotropic for each $\xi \in \mathcal{P} \cup \mathcal{U}$. Then q is isotropic.*

Going further, the patching method is applied to compute the u and strong-u invariant of such fields[23].

[23] Vlerë Mehmeti. Patching over Berkovich curves and quadratic forms. *Compos. Math.*, 155(12):2399–2438, 2019

Definition 6 (Me19)**.** Let K be a field.

1. [Kaplansky] The u-invariant of K, denoted by $u(K)$, is the maximal dimension of anisotropic quadratic forms over K. We say that $u(K) = \infty$ if there exists anisotropic quadratic forms of arbitrarily large dimension.
2. [HHK] The strong u-invariant of K, denoted by $u_s(K)$, is the smallest real number m such that:
   - $u(E) \leq m$ for all finite field extensions E/K;
   - $\frac{1}{2}u(E) \leq m$ for all finitely generated field extensions E/K of transcendence degree 1.

   We say that $u_s(K) = \infty$ if there exists such fields extensions E of arbitrarily large u-invariant.

The computation of u-invariants has been of particular interest in the algebraic theory of quadratic forms[24]. Indeed to paraphrase Izhbodin[25], it is a fact that many questions about quadratic forms over a field K can be reduced to their anisotropic part. In this sense $u(K)$ is a fundamental measure of the complexity of quadratic forms over K. Computing this invariant has been proven very difficult. For instance a conjecture of Kaplansky[26], that held for over 30 years, was that $u(K)$ is always a power of 2. We do not give a counter-example here, but fields with odd u-invariant exists, for an example see ibid.

[24] It is an important step towards the classical Hasse-Minkowski theorem that the u-invariant of p-adic fields is 4.

[25] Oleg T. Izhboldin. Fields of u-invariant 9. *Ann. of Math. (2)*, 154(3):529–587, 2001

[26] The original definition of u-invariant appears in the context of this conjecture made in 1953; no precise reference seem to exist.

The theoretical result that computes u-invariants by the patching technique we have discussed, is as follows[27].

[27] David Harbater, Julia Hartmann, and Daniel Krashen. Applications of patching to quadratic forms and central simple algebras. *Invent. Math.*, 178(2):231–263, 2009

Theorem 7 ([HHK09], 4.10)**.** *Let K be a complete discretely valued field whose residue field k has characteristic unequal to 2. Then $u_s(K) = 2u_s(k)$.*

It leads to the some computations of u-invariants, for p an odd prime and F a one variable function field over K:

- Let K be an m-local field. If k is algebraically closed then $u(F) = 2^{m+1}$. If k is finite then $u(F) = 2^{m+2}$.
- For $K/\mathbf{Q}_p$ a finite extension, one has $u(F) = 8$.
- For $K = \mathbf{Q}_p((t))$, one has $u(F) = 16$.
- If k is algebraically closed or finite, then $u(\mathrm{Frac}(k[[x,t]])) = 4$ and $u(\mathrm{Frac}(k[x][[t]])) = 8$.

The introduction of Berkovic analytification

By using Berkovic analytification, Mehmeti removes the discretely valued hypothesis on the base field k, which previously was the fraction field of the discretely valued ring T. She makes use of the structure of Berkovic spaces by introducing *nice covers* of curves and *parity functions*[28] for such covers[29].

[28] See Definition 2.1 and 2.18 of [Me19] for details.

[29] Vlerë Mehmeti. Patching over Berkovich curves and quadratic forms. *Compos. Math.*, 155(12):2399–2438, 2019

Theorem 8 ([Me19])**.** *Let k be a complete non-trivially valued ultrametric field. Let C be a normal irreducible projective k-algebraic curve. Denote by F the function field of C. Let X be an F-variety and G a connected rational algebraic group acting strongly transitively on X.*

Let $V(F)$ be the set of all non-trivial rank 1 valuations on F which either extend the valuation on k or are trivial when restricted to k.

Denote by C^{an} the Berkovic analytification of C, so that $F = \mathcal{M}(C^{\mathrm{an}})$, where $\mathcal{M}$ denotes the sheaf of meromorphic functions on C^{an}. Then the following local-global principles hold.

- $X(F) \neq \varnothing \iff X(\mathcal{M}_x) \neq \varnothing$ *for all* $x \in C^{\mathrm{an}}$.
- *If F is a perfect field or X is a smooth variety, then*

$$X(F) \neq \varnothing \iff X(F_v) \neq \varnothing \text{ for all } v \in V(F),$$

  where F_v denotes the completion of F with respect to v.

It should be mentioned that Mehmeti first proves the theorem with the added condition that $\sqrt{|k^\times|} = |k^\times| \otimes_{\mathbf{Z}} \mathbf{Q}$ is distinct from $\mathbf{R}_{>0}$. In particular, type 3 points, that is points x on the analytic curve X^{an} such that the rank of the quotient of value groups $|\mathcal{H}(x)^\times|/|k^\times|$ is 1, play an important role[30]. The condition is then removed, which allows for no type 3 points to exist, by using model theory.

[30] This also tells that this method does not apply in the framework of rigid-analytic spaces.

Let us finally remark that this patching theorem recovers Theorem 4 when applied in the same setting. It also provides new bounds for the u-invariants of complete valued non archimedean fields k with rank $n \in \mathbf{N} \setminus \{0\}$ valuations.

References

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Vlerë Mehmeti. Patching over Berkovich curves and quadratic forms. *Compos. Math.*, 155(12):2399–2438, 2019.