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Around the Grothendieck-Teichmüller group

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Introduction

In “Esquisse d’un programme", see [Gro97], Grothendieck motivates the study of the absolute Galois group $G_{\mathbf{Q}}$ of $\mathbf{Q}$ by its action on the tower of the fundamental groups of the moduli stacks $\mathcal{M}_{g,m}$ of curves of genus g with m marked points. The first realization of this program is given by Drinfel’d in [Dri90] using quasitriangular quasi-Hopf algebras in relation with the genus 0 tower, and the construction of a certain group $\widehat{\mathrm{GT}}$, the Grothendieck-Teichmüller group, which is finitely presented, of combinatorial origin, and contains $G_{\mathbf{Q}}$.

At the arithmetic level, a description of $\widehat{\mathrm{GT}}$ as a subgroup of the fundamental group $\pi_1^{et}(\mathbf{P}^1_{\overline{\mathbf{Q}}} \setminus \{0,1,\infty\})$ was then obtained by Ihara in [Iha94] using Belyi’s theorem and explicit tangential basepoints on $\mathcal{M}_{0,4} \simeq \mathbf{P}^1 \setminus \{0,1,\infty\}$, and $\mathcal{M}_{0,5}$. The group $\widehat{\mathrm{GT}}$ is further realized as the outer automorphism group of a $\mathfrak{S}_n$ and inertia invariant genus 0 tower by Harbater-Schneps in [HS00][1]: $\widehat{GT} \simeq \mathrm{Out}^{\sharp}_{\mathfrak{S}_n}(\{\mathcal{T}_{0,n}\}_{n\geq 4})$.

The questions surrounding the Grothendieck-Teichmüller group, such as describing properties (or even characterizing) $G_{\mathbf{Q}}$ by combinatorial means, have seen recent progress and ramifications in diverse mathematical domains. On one side Fresse and Horel have revisited and developed the relation of $\widehat{\mathrm{GT}}$ to algebraic topology via the little disks operads, see [Fre17] for a complete introduction to this construction, see also [BH21] for application to embedding calculus and higher dimensional knot invariants. On an other side, which will be the emphasis of this workshop, Hoshi, Minamide and Mochizuki obtain an anabelian description of $\widehat{GT}$ using combinatorial anabelian geometry in [HMM22] that shows new relations to the étale fundamental groups of configuration spaces of hyperbolic curves. One of the result of this approach is the surprising isomorphism

$$\widehat{\mathrm{GT}} \times \mathfrak{S}_5 \simeq \mathrm{Out}\,\Pi_2$$

where Π_2 is the étale fundamental group of the second configuration space of $\mathbf{P}^1 \setminus \{0,1,\infty\}$. This approach, completed by the use of combinatorial Belyi cuspidalization, has furthermore recently led to a certain (class of) subgroup BGT, potentially isomorphic to $\widehat{GT}$, which provides a combinatorial model $\overline{\mathbb{Q}}_{\mathrm{BGT}}$ of the (class of) algebraic closure of rational numbers, see [HMT20]. For applications to the outer automorphisms of braid groups we refer to Minamide and Nakamura in [MN22].

The goal of the workshop is to provide an arithmetical construction of the Grothendieck-Teichmüller group such as given by Ihara and a practical understanding of the anabelian and geometric methods, relying on log compactification, used by Hoshi, Minamide and Mochizuki to provide recent results on $\widehat{GT}$. The main references for the talks will be [Iha94] and [HMM22].

[1] For further studies in Grothendieck-Teichmüller theory we refer to the work of Enriquez, Lochak, Nakamura and Schneps.

This workshop is part of the France-Japan Arithmetic and Homotopic Galois Theory RIMS-CNRS international research network.[2]

PROGRAM

Talk 1 – The classical Grothendieck-Teichmüller group (L. Dauger) (1 hour)

The goal of this talk is to present the arithmetical construction of the Grothendieck-Teichmüller group $\widehat{GT}$.
The Grothendieck-Teichmüller group will be introduced as the group associated to the 2-parameters set $\{(\lambda, f) \in \widehat{\mathbf{Z}} \times \widehat{F}_2\}$ satisfying the 3 relations I/ II/ and III/ (coming from $\mathcal{M}_{0,4}$ and $\mathcal{M}_{0,5}$ respectively). The speaker will first recall the existence of the homotopic exact sequence for variety over $\mathbf{Q}$ and the resulting outer Galois action. Up to the speaker's choice a quick proof of injectivity in the case of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ as a consequence of Belyi's theorem can be given. The lift of this outer action as a map $G_{\mathbf{Q}} \to \widehat{\mathbf{Z}} \times \widehat{F}_2$ by the use of tangential basepoints following Ihara will be detailed as well as the resulting relations I/ and II/ defining $\widehat{GT}$, see appendix of [EL94].

References: [Iha94], [EL94].

Talk 2 – Log geometry and compactification of $\mathcal{M}_{g,m}$ (N. Takada) (1 hour)

This talk should serves as an introduction to log geometry and log schemes with the specific goal of showing that the moduli space for log stable curves gives a natural compactification of $\mathcal{M}_{g,m}$. The notion of logarithmic structure on a scheme should be presented (for example : the main parts of section 1 of [Kat89] and the definition of smoothness/étaleness of log schemes as in section 3) with standard examples up to the speaker's choice followed by the description of the log structure on the compactification $\overline{\mathcal{M}_{g,m}}$ of $\mathcal{M}_{g,m}$. The Proposition 1.1 of [Kat00] giving an étale local description of log stable curves and its translation as a functor of stacks as (4.1) of *loc cit.* will be explained. Some details on Theorem 4.1 giving the isomorphism between $\overline{\mathcal{M}_{g,m}}$ with its canonical log structure and the moduli space of stable log curves $\mathcal{LM}_{g,n}$ of type (g, n) up to the speakers choice should be given.

References: [Kat89], [Kat00].

Talk 3 – Some techniques of anabelian geometry (A. Assoun) (1 hour)

In this talk, the speaker will discuss some group-theoretical reconstruction results in anabelian geometry used by Hoshi, Minamide and Mochizuki to prove $\mathrm{GT}^{\Sigma} \times \mathfrak{S}_5 \simeq \mathrm{Out}(\Pi_2)$ in [HMM22], which is the main objective of this workshop. More precisely, the speaker will explain the group-theoretical reconstruction of the geometric invariants such as the genus g, the number of cusps r and the dimension n: the reconstruction process of the dimension n is the content of Theorem 1.6 of [HMM22]; the reconstruction of g and r will focus on the case $r \geq 1$ of Theorem 2.5 (vi-d) of [HMM22] and the method of the proof of [Moc04] Lemma 1.3.9, see also [Tam97] Proposition 3.5 (the sorting criterion by quasi-co-surface subgroups properties of Proposition 2.3 of [HMM22] can be ignored). The speaker can also mention for "the group-theoretical reconstructions of

[2] Homepage for the workshop: https://ahgt.math.cnrs.fr/activities/ateliers/AGA24-around%20GT/, February 6, 2024

inertia groups", e.g. [Nak90] Theorem 3.4 and [HMM22] Section 4.

References: [HMM22], [Moc04], [Nak90], [Tam97].

Talk 4 – An anabelian description of $\widehat{\mathrm{GT}}$ (N. Yamaguchi) (1 hour 30 minutes)

In this talk the speaker should first present the anabelian description of $\widehat{\mathrm{GT}}$ as the FCS-admissible outer automorphisms of the 2 configuration space given by Definition 2.7 of [HMM22]. Then, introducing the different notion of F, FC and gF-admissibility and their relation to FCS-admissibility the proof of the isomorphism $\mathrm{GT}^{\Sigma} \times \mathfrak{S}_{n+3} \simeq \mathrm{Out}(\Pi_n)$ should be sketched; the required intermediate results will be explicitly stated. In the proof, the speaker need to mention the group-theoretical reconstruction of the generalized surface subgroups, treated in Theorem 2.5 (iv), (v), and introduce the proof if times allows. The group-theoretical reconstruction algorithms for $\widehat{\mathrm{GT}}$ and $\mathfrak{S}_n^*$ should also be introduced (cf. [HMM22] Corollaries 2.8-2.10). Optionally, further results on $\widehat{\mathrm{GT}}$ developed in [HMT20] can be introduced (in particular Theorem G).

References: [HMM22], [HMT20].

References

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[Dri90] V. G. Drinfel'd. "On quasitriangular quasi-Hopf algebras and on a group that is closely connected with $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$". In: *Algebra i Analiz* 2.4 (1990), pp. 149–181.

[EL94] Michel Emsalem and Pierre Lochak. "The action of the absolute Galois group on the moduli space of spheres with four marked points". In: *The Grothendieck theory of dessins d'enfants (Luminy, 1993)*. Vol. 200. London Math. Soc. Lecture Note Ser. Appendix of : On the embedding of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ into $\widehat{\mathrm{GT}}$ by Y. Ihara. Cambridge Univ. Press, Cambridge, 1994, pp. 307–321.

[Fre17] Benoit Fresse. *Homotopy of operads and Grothendieck-Teichmüller groups. Part 2*. Vol. 217. Mathematical Surveys and Monographs. The applications of (rational) homotopy theory methods. American Mathematical Society, Providence, RI, 2017, pp. xxxv+704.

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[HMT20] Yuichiro Hoshi, Shinichi Mochizuki, and Shota Tsujimura. "Combinatorial Construction of the Absolute Galois Group of the Field of Rational Numbers". In: *RIMS preprint* 1935 (2020), p. 98.

[Iha94] Yasutaka Ihara. "On the embedding of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ into $\widehat{\mathrm{GT}}$". In: *The Grothendieck theory of dessins d'enfants (Luminy, 1993)*. Vol. 200. London Math. Soc. Lecture Note Ser. With an appendix: the action of the absolute Galois group on the moduli space of spheres with four marked points by Michel Emsalem and Pierre Lochak. Cambridge Univ. Press, Cambridge, 1994, pp. 289–321.

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[MN22] Arata Minamide and Hiroaki Nakamura. "The automorphism groups of the profinite braid groups". In: *Amer. J. Math.* 144.5 (2022), pp. 1159–1176.

[Moc04] Shinichi Mochizuki. "The absolute anabelian geometry of hyperbolic curves". In: *Galois theory and modular forms*. Vol. 11. Dev. Math. Kluwer Acad. Publ., Boston, MA, 2004, pp. 77–122.

[Nak90] Hiroaki Nakamura. "Galois rigidity of the étale fundamental groups of punctured projective lines". In: *J. Reine Angew. Math.* 411 (1990), pp. 205–216.

[Tam97] Akio Tamagawa. "The Grothendieck conjecture for affine curves". In: *Compositio Math.* 109.2 (1997), pp. 135–194.

Schedule - France

The workshop takes place at ENS PSL (Room R3 of the department of mathematics and applications).

8:00	*Meeting at Jussieu*
8:30 - 9:30	Talk 1: The classical Grothendieck-Teichmüller group.
9:30 - 10:30	Talk 2: Log geometry and compactification of $\mathcal{M}_{g,m}$.
10:30 - 11:00	*Break*
11:00 - 12:00	Talk 3: Some techniques of anabelian geometry.
12:00 - 13:00	*Lunch Break*
13:00 - 14:30	Talk 4: An anabelian description of $\widehat{\mathrm{GT}}$.

Schedule - Japan

The workshop takes place at RIMS (room 110).

16:00	*Meeting at RIMS*
16:30 - 17:30	Talk 1: The classical Grothendieck-Teichmüller group.
17:30 - 18:30	Talk 2: Log geometry and compactification of $\mathcal{M}_{g,m}$.
18:30 - 19:00	*Break*
19:00 - 20:00	Talk 3: Some techniques of anabelian geometry.
20:00 - 21:00	*Dinner Break*
21:00 - 22:30	Talk 4: An anabelian description of $\widehat{\mathrm{GT}}$.