Atelier de Géométrie Arithmétique
数論幾何学のアトリエ

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Spaces and perfectoids towards a perfectoid Siegel modular space

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Location and time: RIMS Kyoto, Japan and Paris, France - Summer 2024

Introduction

The notion of a perfectoid space was introduced by Scholze in [Sch12]. It readily had a plethora of strong applications such as in p-adic Hodge theory [BMS18] and the Langlands program [Sch15]. To quote Bhatt in [Bha14] "The theory of perfectoid spaces is rather young, but already extremely potent: each class of examples discovered so far has led to powerful and deep theorems in arithmetic geometry".

The inspiration of perfectoid spaces come from the reunion of a classical result of Fontaine and Wintenberger in Galois theory, the theories of analytic geometry over non-archimedean spaces as given by Huber adic spaces, Berkovich spaces and the rigid-analytic geometry of Tate, and Faltings "almost mathematics" developped by Gaber and Ramero in [GR03].

As a first notion of perfectoids we should consider perfectoid fields. These are non-discretely valued characteristic 0 fields K for which there is a characteristic p counterpart $K^\flat$, called its *tilt*, such that the absolute Galois groups of K and $K^\flat$ are naturally isomorphic through the tilting operation. For example, the Fontaine-Wintenberger isomorphism is as follows

$$\mathrm{Gal}\left(\widehat{\mathbf{Q}_p(p^{\frac{1}{p^\infty}})}\right) \cong \mathrm{Gal}\left(\widehat{\mathbf{F}_p((t))(t^{\frac{1}{p^\infty}})}\right).$$

Perfectoid spaces are a higher dimensional version of such objects. They can be seen as giving a bridge between the theory of algebraic varieties over K and its tilt $K^\flat$. This idea can be represented by the following diagram ([Bha14], p.1084)

$$\begin{array}{ccccc} \text{char.}\, K = 0 & \mathrm{Var}/K & & \mathrm{Var}/K^\flat & \text{char.}\, K^\flat = p \\ & \downarrow & & \downarrow {\scriptstyle perfection} & \\ & \mathrm{Perf}/K & \underset{untilt}{\overset{tilt}{\longleftrightarrow}} & \mathrm{Perf}/K^\flat & \end{array}$$

Figure 1: Perfectoid spaces as a bridge between characteristics

The horizontal *tilting* morphisms are given by two fundamental theorems of Scholze which asserts that

- the category of perfectoid spaces over K and the category of perfectoid spaces over $K^\flat$ are equivalent;

[1]Homepage for the workshop: https://ahgt.math.cnrs.fr/activities/ateliers/AGA24-spaces%20perfectoid, program version of May 3, 2024.

- for a perfectoid space X over K the tilting operation induces an equivalence of Grothendieck sites $X_{et} \simeq X^{\flat}_{et}$.

One should note that the dashed mapping of Fig. 1 is only well-defined under the existence, at the level of some relevant arithmetic geometry space, of a so-called "perfection". One striking example, shown by Scholze in [Sch15], is the case of the moduli space of abelian varieties with a trivialisation of their p^∞-torsion. This leads further to the construction of a Hodge-Tate period map whose ongoing study continues to provide leading results such as in [CS24].

The goal of the workshop is to provide an introduction to the theory of perfectoid spaces and see it in application with the construction of a Siegel perfectoid space and a Hodge-Tate period map following [Sch12] and [Sch15]. The classical GIT-construction of a fine moduli space for abelian varieties with level structure will be presented.

This workshop is part of the France-Japan Arithmetic and Homotopic Galois Theory RIMS-CNRS international research network.[1]

PROGRAM

Talk 1 – Perfectoid fields and rings. (E. Caeiro) (1 hour)

This talk will start by introducing the notion of perfectoid fields (cf. [MorM2], p.2) with the first basic example given by Fontaine and Wintenberger explained before moving to the more general setting of perfectoid rings and algebras.
In this talk, the speaker *aims to explain Theorem 1.11 and Theorem 4.5 of [MorM2]* (Theorem 3.7 of [Sch12]). To that end, the speaker will outline the definitions of *integral perfectoid rings*, and *perfectoid Tate rings* (cf. [MorM2], Definition 3.1 and p.21) and discuss their relationships. Some examples from [Bha14] should be treated. Next, the speaker will explain the brief proof of the tilting correspondence (cf. [MorM2], Theorem 1.11) for perfectoid algebras, as detailed in section 1. Finally, the speaker will introduce the *correspondence of étale sites* (cf. [MorM2], Theorem 4.5 ([Sch12], Theorem 3.7)), referring to the necessary content from the sections 2-4 of [MorM2] ([Sch12], sections 1-5).

It is crucial that the speaker mentions the *almost purity theorem for perfectoid algebras* (cf. [Sch12], Propositions 5.22 and 5.25), which is used in the proof of [MorM2] Theorem 4.5, stating both theorems and only giving a short sketch of the proofs if time permits.

References: [Sch12], [MorM2] and [Bha14].

Talk 2 – Moduli space of abelian varieties with level structure (A. Klughertz) (1 hour 30 minutes)

In this talk the speaker will *introduce the different elements necessary to define the Siegel modular space* $\mathcal{A}_{g,d}[n]$ with level-n structure as used in [Sch15] in preparation for Talk 4.
First, *abelian* varieties and their *polarizations* should be introduced, for instance as in [EGM24]. Going further the speaker will present the construction of the ℓ-adic Tate modules $\mathrm{T}_\ell\, A$ and the associated Galois representations (cf. [EGM24] chapter 10). Specifying to the case of a p-adic base field K and good reduction, the speaker will introduce the relation between $\mathrm{T}_p\, A$ and the p-divisible group $\mathscr{A}[p^\infty]$ over the ring of integers $\mathscr{O}_K$. At this point the *Hodge-Tate filtration*

Lie $A \subset K^{2g}$ should be introduced as in Proposition 3.3.1 of [Sch15], noting that the filtration descends to K as mentioned in Lemma 3.3.4 of *loc.cit.*
The rest of the talk should present *the construction of* $\mathcal{A}_{g,d}[n]$, following [GIT] chapter 6 and 7. First the notion of an abelian scheme with level-n structure should be given as in Definition 7.2 of [GIT] followed by the definition of $\mathcal{A}_{g,d}[n]$ as a functor over the category of schemes. The definitions of *fine and coarse moduli scheme* should be introduced. The steps of the construction of $\mathcal{A}_{g,d}[n]$ as the quotient of $H_{g,d,n}$ (cf. Definition 7.6) will be sketched (introduction of the Hilbert scheme as in p.131-132, $H_{g,d,n}$ as a subscheme cf. Proposition 7.3 and finally the quotient step of Proposition 7.6).

References: [EGM24], [GIT].

Talk 3 – Perfectoid spaces, tilts and untilts (R. Ishizuka) (1 hour 30 minutes)

This talk will provide *the main result* of [Sch12], all references for this talk are from *loc. cit.*.
We will first go over the theory of *adic spaces* (or equivalently *Berkovich spaces* up to the speakers choice) in order to define affinoid perfectoid spaces and *perfectoid spaces*. The speaker aims to explain Theorem 7.12, focusing on the contents of sections 6-7. First, the speaker will mention the definition of the perfectoid affinoid k-algebra (cf. Definition 6.1) and the *titling correspondence* of the affinoid k-algebra (cf. Lemma 6.2), with a possible reference to Talk 1. Next, the speaker will present Theorem 6.3, focusing on (iii), and, using this result, define *perfectoid spaces* as in Definition 6.15. The speaker will then introduce the proof of the *almost purity theorem* for perfectoid spaces (cf. Theorem 7.9), referring to the necessary content from sections 6-7.
Finally, the proof of the *correspondence of the étale sites* Theorem 7.12 will be discussed. Proofs in this talk should be a short sketch up and details are up to the speakers choice.

References: [Sch12].

Talk 4 – A perfectoid Siegel modular space and a Hodge-Tate period map (S. Philip) (1 hour 30 minutes)

The goal of this talk is to present Theorem 3.1.2 of [Sch15], that is the construction of a perfectoid Siegel modular space with its Hodge-Tate period map.
The speaker will present the *integral theory of canonical subgroups* for ordinary abelian varieties and how it provides the way to construct a perfection of the Siegel moduli space. This should be done by providing Lemma 3.1.3 with some ideas of proof as in the first part of Section 3.2. The introduction of $\mathfrak{X}^*(\varepsilon)$ should be done with the aim of Lemma 3.2.17 in mind. The speaker will present the *local tilting result* of Corollary 3.2.19 as it is an important step of producing the dashed arrow of Fig. 1 in this context. The different steps of the proof of Theorem 3.2.36 will be summarized and finally Corollary 3.3.12 with its proof should be presented.
The first definition of the *Hodge-Tate period map* as in Lemma 3.3.4 by means of the Hodge-Tate filtration should be given and a description on the *locus of good reduction* should be mentioned (cf. the introduction of Section 3.).

References: [Sch15].

References

[Bha14] B. Bhatt. "What is . . . a perfectoid space?" In: *Notices Amer. Math. Soc.* 61.9 (2014), pp. 1082–1084.

[BMS18] B. Bhatt, M. Morrow, and P. Scholze. "Integral p-adic Hodge theory". In: *Publ. Math. Inst. Hautes Études Sci.* 128 (2018), pp. 219–397.

[CS24] A. Caraiani and P. Scholze. "On the generic part of the cohomology of non-compact unitary Shimura varieties". In: *Ann. of Math. (2)* 199.2 (2024), pp. 483–590.

[EGM24] B. Edixhoven, G. van der Geer, and B. Moonen. *Abelian varieties*. Preliminary version. 2024, p. 331.

[GR03] O. Gabber and L. Ramero. *Almost ring theory*. Vol. 1800. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2003, pp. vi+307.

[MorM2] M. Morrow. *Adic and perfectoid spaces*. IMJ–PRG specialised M2 course. 2018. URL: https://webusers.imj-prg.fr/~matthew.morrow/Espaces/Perf%20spaces%20IMJ-PRG%20final.pdf.

[GIT] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*. Third. Vol. 34. Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]. Springer-Verlag, Berlin, 1994, pp. xiv+292.

[Sch12] P. Scholze. "Perfectoid spaces". In: *Publ. Math. Inst. Hautes Études Sci.* 116 (2012), pp. 245–313.

[Sch15] P. Scholze. "On torsion in the cohomology of locally symmetric varieties". In: *Ann. of Math. (2)* 182.3 (2015), pp. 945–1066.

Schedule - France

Exceptionnally the workshop will take place online via Zoom.

8:00	*Meeting online*
8:30 - 10:00	Talk 1: Perfectoid fields and rings
10:15 - 11:45	Talk 2: Moduli space of abelian varieties with level structure
11:45 - 12:45	*Break*
13:00 - 14:30	Talk 3:
14:30 - 15:30	Talk 4: A perfectoid Siegel modular space and a Hodge-Tate period map

Schedule - Japan

The workshop takes place at RIMS, in room 110.

15:00	*Meeting at RIMS*
15:30 - 17:00	Talk 1:
17:15 - 18:45	Talk 2:
18:45 - 19:45	*Break*
20:00 - 21:30	Talk 3: Perfectoid spaces, tilts and untilts
21:30 - 22:30	Talk 4: A perfectoid Siegel modular space and a Hodge-Tate period map