



## Local-global principles and the patching method

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### INTRODUCTION

The patching method has a long history in inverse Galois theory going back to applications of Serre's GAGA principle to algebraize complex patched coverings. Later, Harbater proved, by patching and gluing  $G$ -covers with totally  $K$ -rational fibers with  $K$  a complete valued field, the Regular Inverse Galois Problem for such fields in [Har87] and the Abhyankar's conjecture in [Har94]. This method has been brought to a new light by Harbater, Hartmann and Krashen in [HH10] and [HHK09] where they develop patching techniques over function fields of algebraic curves over complete discretely valued fields.

A key element in solving patching problems over fields is given by factorisation in linear algebraic groups. For example, given fields  $F_1, F_2 \subset F_0$  the factorization of all  $A \in GL_n(F_0)$  into  $A_1 A_2$  with  $A_i \in GL_n(F_i)$  is equivalent to the patching problem over those fields. Solving these factorization problems using the properties of complete local function fields of curves – such as the reduction map to the special fiber – is one of the main technical parts. In recent years Mehmeti has improved these patching methods by using Berkovich analytification in [Meh19] allowing her to work with more general fields, i.e. removing the condition of the valuation being discrete.

These techniques result in local-global theorems with various arithmetic applications such as computing  $u$ -invariants – an integer giving the maximal dimension of anisotropic quadratic forms – of such fields. As an example it is computed that  $u(\mathbf{Q}_p((T))) = 16$  for odd primes  $p$ .

The goal of the workshop is to give a practical understanding of the patching method and how it provides arithmetic results. The main references for the talks will be [Ber90], [HHK09], [HH10] and [Meh19].

This workshop is part of the France-Japan [Arithmetic and Homotopic Galois Theory](#) RIMS-CNRS international research network.<sup>1</sup>

### PROGRAMME

**Talk 1 – Patching problems and local-global results. (S. Philip)** (50 minutes)

The goal of this talk is to present the local-global results of [HH10], [HHK09] and [Meh19] and the patching method.

The talk will provide a formal definition of a patching problem and solution starting with the example of the categories of vector spaces over fields (see section 2 of [HH10]). The speaker

<sup>1</sup>homepage for the workshop: <https://ahgt.math.cnrs.fr/activities/ateliers/AGA23-patching/>, June 21, 2023

will then explain local-global principles and the results of the references in that context. The rest of the talk will be devoted to specify the patching problems that arise in the proofs of these results and how they are approached using homogeneous spaces under linear algebraic groups (without providing technical proofs).

*References:* [HH10], [HHK09] and [Meh19].

**Talk 2 – Arithmetic applications. (L. Loiseau)**

(1 hour 15 minutes)

With this talk we will investigate what kind of arithmetic results the patching method and local-global principles given by Harbater, Hardmann, Krashen and Mehmeti produce. To that end, introduce the  $u$ -invariant and strong  $u$ -invariant of fields and the results obtained on those in [HHK09] and [Meh19]. If the speaker wants the results on the cohomological dimension of fields can also be given and explained. The speaker will show how the local-global results obtained by Harbater, Hartmann, Krashen and Mehmeti give ways to compute those in the case of complete discretely valued fields by reducing to the local case. Precisely, explain the reduction to a local problem (cf [HHK09] section 4 or 5) using the local-global principle and how the local problem is dealt with. The speaker can choose to deal with central simple algebras or quadratic forms. Some illustrations of the results are welcome.

*References:* [HHK09] and [Meh19].

**Talk 3 – Patching and linear algebraic groups. (N. Yamaguchi)**

(1 hour 15 minutes)

The talk will explain the technical part of how homogeneous spaces under rational connected linear algebraic groups are used to make the patching method work.

The speaker will explain how the patching problems that arise (described in 2.1) are solved by factorization results for linear algebraic groups starting with a proof in the case of fields as in [Har84] Proposition 2.1. The factorization result in the case of  $GL_n$  for discretely valued complete fields ([HH10] section 5.1) will be given with an overview of the intersection part in order to present theorem 5.8 of loc. cit. Then theorem 5.9's proof will be explained (with the results of global patching only presented without proof). An example to illustrate the result shall be given (see p.85–86 of [HH10] for instance).

*References:* [Har84] and [HH10].

**Talk 4 – The introduction of Berkovich spaces. (K. Goto)**

(1 hour 15 minutes)

Starting with an introduction to Berkovich spaces and Berkovich analytification using [Ber90] (and/ or [Tem10]) the aim of the talk is to show how patching can be applied to analytic curves. The introduction should focus on examples and illustrations (for instance the affine and projective lines) and only give the important definitions of the first chapters of [Ber90]. The different types of points on analytic curves (see [Tem10] Definition 2.3.3.3) and the importance of type 3 points in Mehmeti's construction should be explained. The results on the existence of *nice* covers should be given but not necessarily detailed (up to the speaker's choice). The comparison between the results of [HHK09] and [Meh19] (see [Meh19] part 4) will be detailed in this talk.

*References:* [Ber90], [Tem10] and [Meh19].

## REFERENCES

- [Ber90] Vladimir G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*. English. Vol. 33. Math. Surv. Monogr. Providence, RI: American Mathematical Society, 1990. ISBN: 0-8218-1534-2.
- [Har84] David Harbater. “Convergent arithmetic power series”. English. In: *Am. J. Math.* 106 (1984), pp. 801–846. ISSN: 0002-9327. DOI: [10.2307/2374325](https://doi.org/10.2307/2374325).
- [Har87] David Harbater. *Galois coverings of the arithmetic line*. English. Number theory, Semin. New York 1984/85, Lect. Notes Math. 1240, 165-195 (1987). 1987.
- [Har94] David Harbater. “Abhyankar’s conjecture on Galois groups over curves”. English. In: *Invent. Math.* 117.1 (1994), pp. 1–25. ISSN: 0020-9910. DOI: [10.1007/BF01232232](https://doi.org/10.1007/BF01232232).
- [HH10] David Harbater and Julia Hartmann. “Patching over fields”. English. In: *Isr. J. Math.* 176 (2010), pp. 61–107. ISSN: 0021-2172. DOI: [10.1007/s11856-010-0021-1](https://doi.org/10.1007/s11856-010-0021-1).
- [HHK09] David Harbater, Julia Hartmann, and Daniel Krashen. “Applications of patching to quadratic forms and central simple algebras”. English. In: *Invent. Math.* 178.2 (2009), pp. 231–263. ISSN: 0020-9910. DOI: [10.1007/s00222-009-0195-5](https://doi.org/10.1007/s00222-009-0195-5).
- [Meh19] Vlerë Mehmeti. “Patching over Berkovich curves and quadratic forms”. English. In: *Compos. Math.* 155.12 (2019), pp. 2399–2438. ISSN: 0010-437X. DOI: [10.1112/S0010437X19007632](https://doi.org/10.1112/S0010437X19007632).
- [Tem10] Michael Temkin. *Introduction to Berkovich analytic spaces*. 2010. DOI: [10.48550/ARXIV.1010.2235](https://doi.org/10.48550/ARXIV.1010.2235). URL: <https://arxiv.org/abs/1010.2235>.

## SCHEDULE - FRANCE

The workshop takes place at Sorbonne Université (Jussieu campus, room 1516-411, 4th floor, corridor 1516, access via towers 25 or 26).

- 8:00            *Meeting at Jussieu*
- 8:30 - 9:30    Talk 1: Patching problems and local-global results
- 9:30 - 11:00   Talk 2: Arithmetic applications
- 11:00 - 11:30   *Break*
- 11:30 - 13:00   Talk 3: Patching and linear algebraic groups
- 13:00 - 14:00   *Lunch Break*
- 14:00 - 15:30   Talk 4: The introduction of Berkovich spaces.

## SCHEDULE - JAPAN

The workshop takes place at RIMS (Botanic Annex room 003).

- 15:00            *Meeting at RIMS*
- 15:30 - 16:30   Talk 1: Patching problems and local-global results
- 16:30 - 18:00   Talk 2: Arithmetic applications
- 18:00 - 18:30   *Break*
- 18:30 - 20:00   Talk 3: Patching and linear algebraic groups
- 20:00 - 21:00   *Dinner Break*
- 21:00 - 22:30   Talk 4: The introduction of Berkovich spaces.