Atelier de Géométrie Arithmétique
数論幾何学のアトリエ

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Étale homotopy theory and application

Organizers: M. Ferreira-Filoramo (IMJ-PRG) & A. Galet (IMJ-PRG) & N. Yamaguchi (Institute of Science Tokyo)

Location and time: RIMS Kyoto, Japan and Paris, France - 26 September 2025

INTRODUCTION

Étale homotopy theory of schemes, first introduced by M. Artin and B. Mazur [AM69] in the 1960s, is an analogue in algebraic geometry of classical homotopy theory for topological spaces. Roughly speaking, for any scheme X, one considers the category $\mathbf{HR}(X)$ of all (étale) hypercoverings of X and constructs a functor

$$\mathbf{HR}(X) \to \mathbf{Set}^{\Delta^{op}}$$

called a Verdier functor. By passing to the homotopy category $\mathrm{Ho}(\mathbf{HR}(X))$ of $\mathbf{HR}(X)$, one obtains a pro-simplicial set of X, which is called the étale homotopy type of X.

A problem with Artin-Mazur's construction is that $\mathbf{HR}(X)$ is not cofiltered, and so we have to take the homotopy category. This issue was fixed by introducing rigid hypercoverings $\mathbf{HHR}(X)$, which refine the original Artin-Mazur étale homotopy theory, and defining the étale topological type $X_{\text{ét}}$ of X (cf. [Fri82]). Étale homotopy theory has continued to be used in recent years; see, for example, [HHW24].

A promising application of étale homotopy theory was initiated in [SS16], that is, for k a finitely generated field over $\mathbb{Q}$ and X, Y hyperbolic curves or strongly hyperbolic Artin neighborhoods over k, the map

$$\mathrm{Isom}_k(X, Y) \longrightarrow \mathrm{Isom}_{\mathrm{Ho(pro\text{-}ss)}\downarrow k_{\text{ét}}}(X_{\text{ét}}, Y_{\text{ét}})$$

is bijective. In other words, two hyperbolic curves are k-isomorphic if and only if their étale topology types are isomorphic in $\mathrm{Ho(pro\text{-}ss)} \downarrow k_{\text{ét}}$. The fact that the first homotopy group of $X_{\text{ét}}$ coincides with Grothendieck's étale fundamental group implies that this result can be viewed as a reinterpretation of Grothendieck's anabelian conjecture that was proved by A. Tamagawa [Tam97] and S. Mochizuki [Moc99]. A similar higher dimensional result was obtained by Y. Hoshi with different techniques, see [Hos20].

In the workshop, we will study the main constructions of the étale topological type $X_{\text{ét}}$ of X and tools of étale homotopy theory with illustrative examples. We will also cover the definition of the étale homotopy groups $\pi_n^{\text{ét}}(X)$. The aim of the workshop is to present anabelian geometry through the lens of étale topology types, as developed in [SS16].

This workshop is part of the France-Japan Arithmetic and Homotopic Galois Theory RIMS-CNRS international research network.[1]

[1] homepage for the workshop: https://ahgt.math.cnrs.fr/activities/ateliers/AGA25-etale_homotopy/

Programme

Talk 1 – Simplicial schemes and their cohomologies (50 minutes)

In this talk, the speaker will aim to explain the contents of [Fri82, Sections 1, 2]. The talk will begin with an introduction to the notion of (bi-)simplicial schemes with some basic examples in [Fri82, Section 1]. To this end, the speaker will explain the definition of the étale site $\acute{\mathrm{E}}\mathrm{t}(X)$ of the simplicial scheme X and the abelian sheaves on $\acute{\mathrm{E}}\mathrm{t}(X)$ (cf. [Fri82, Definition 2.1]). These definitions naturally lead to the cohomology groups of the simplicial scheme X. Finally, the speaker will explain the relation between usual and simplicial cohomologies (cf. [Fri82, Proposition 2.4]). For all propositions only a short sketch of the proof will be given, not a complete proof.

References: [Fri82, Sections 1, 2] .

Talk 2 – Definition of the étale topology type of a simplicial scheme (50 minutes)

In this talk the speaker will introduce the contents of [Fri82, Sections 3, 4] and aim to explain the definition of the étale topology type $X_{\acute{e}t}$ of a simplicial scheme X (cf. [Fri82, Definition 4.4]). First, the speaker will explain the definition of hypercoverings of X (cf. [Fri82, Definition 3.3]). Some properties of their cohomologies will also be introduced, focusing on the relation between sheaf cohomology and hypercoverings (cf. [Fri82, Theorem 3.8]). Next, the speaker will explain the definition of a rigid hypercovering of X (cf. [Fri82, Definition 4.4]) and discuss how it differs from general hypercoverings. Finally, from [Fri82, Section 4], the speaker will selectively present the material that the speaker considers important. The content from [Fri82, Sections 1, 2] are already introduced in Talk 1, the speaker can use it without proof. For all propositions, only a short sketch of proof will be given, not a full proof. Some concrete examples should be added to this talk so that the étale topological type is not just an abstract concept for the audience, but something one can work with.

References: [Fri82, Sections 3, 4] .

Talk 3 – The étale homotopy/cohomology groups of an étale topology type (50 minutes)

The aim of this talk is to introduce [Fri82, Section 5], which defines the étale homotopy groups of $X_{\acute{e}t}$, and the étale cohomology groups of $X_{\acute{e}t}$ with abelian local coefficients, and compares them with classical objects. The speaker will first present the definition of the étale homotopy/cohomology groups of $X_{\acute{e}t}$ (cf. [Fri82, Definition 5.1]) and their relationship to Grothendieck's étale fundamental group. Finally, the speaker will explain the comparison between the cohomology groups of a simplicial scheme and that of its étale topology type. The content from [Fri82, Sections 1-4] is already introduced, the speaker can use it without proof.

References: [Fri82, Section 5]

Talk 4 – Introduction to étale topological anabelian results (50 minutes)

This talk is intended to provide an overview of the main result [SS16, Theorem 1.2] and will not go into the details of the proofs. The speaker will introduce the paper [SS16] in the context of classical anabelian geometry results (cf. [NTM01]). In particular, the speaker should refer to the result of A. Tamagawa [Tam97, Theorem 0.4] and S. Mochizuki [Moc99, Theorem A]. Moreover, the techniques for the Lefschetz trace formula in anabelian geometry developed by A. Tamagawa (cf. [Tam97, Corollary 2.10]) may also be mentioned. Using the comparison between homotopy groups as étale topological types and Grothendieck's fundamental groups prepared in Talks 1–3, the speaker will explain how [SS16] reinterprets classical anabelian geometry as in [Moc99, Theorem A] (via [SS16, Propositions 2.4, A.16], for example). If time permits, the speaker will explain the notion of Artin neighborhood [SS16, Definition 6.1], give an exposition of [SS16, Corollary 1.7], and mention the strong retraction [SS16, Theorem 4.7] which is one of the original contributions of their work.

References: [SS16], [Tam97], [Moc99] [NTM01]

References

[AM69] M. Artin and B. Mazur. *Etale homotopy*. Vol. No. 100. Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1969, pp. iii+169.

[Fri82] Eric M. Friedlander. *Étale homotopy of simplicial schemes*. Vol. No. 104. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1982, pp. vii+190.

[HHW24] Peter J. Haine, Tim Holzschuh, and Sebastian Wolf. "Nonabelian basechange theorems and étale homotopy theory". In: *J. Topol.* 17.4 (2024), Paper No. e70009, 45. DOI: 10.1112/topo.70009.

[Hos20] Yuichiro Hoshi. "A note on an anabelian open basis for a smooth variety". In: *Tohoku Math. J. (2)* 72.4 (2020), pp. 537–550. DOI: 10.2748/tmj.20190917a.

[Moc99] Shinichi Mochizuki. "The local pro-p anabelian geometry of curves". In: *Invent. Math.* 138.2 (1999), pp. 319–423. DOI: 10.1007/s002220050381.

[NTM01] Hiroaki Nakamura, Akio Tamagawa, and Shinichi Mochizuki. "The Grothendieck conjecture on the fundamental groups of algebraic curves". In: *Sugaku Expositions* 14.1 (2001). Sugaku Expositions, pp. 31–53.

[SS16] Alexander Schmidt and Jakob Stix. "Anabelian geometry with étale homotopy types". In: *Ann. of Math. (2)* 184.3 (2016), pp. 817–868. DOI: 10.4007/annals.2016.184.3.5.

[Tam97] Akio Tamagawa. "The Grothendieck conjecture for affine curves". In: *Compositio Math.* 109.2 (1997), pp. 135–194. DOI: 10.1023/A:1000114400142.

Schedule

The workshop takes place at TBA

France

8:30	*Meeting at TBA*
9:00 - 10:00	Talk 1:
10:00 - 11:00	Talk 2:
11:00 - 12:00	*Lunch Break*
12:00 - 13:00	Talk 3:
13:00 - 14:00	Talk 4:

Japan

15:30	*Meeting at RIMS in the lobby*
16:00 - 17:00	Talk 1:
17:00 - 18:00	Talk 2:
18:00 - 19:00	*Dinner Break*
19:00 - 20:00	Talk 3:
20:00 - 21:00	Talk 4: